L I a KUMON

Logarithmic Functions

$a^n = N$ can be written as $\log_a N = n$.

$\log_2 8$ is read as: "the logarithm of 8 to the base 2".

1. Rewrite each of the following expressions in the form $\log_a N = n$.

Ex.
$2^3 = 8 \Rightarrow \log_2 8 = 3, \quad 2^{-3} = \frac{1}{8} \Rightarrow \log_2 \frac{1}{8} = -3$

(1) $2^4 = 16 \Rightarrow \mathbf{\log_2 16 = 4}$

(2) $5^3 = 125 \Rightarrow \mathbf{\log_5 125 = 3}$

(3) $3^{-2} = \frac{1}{9} \Rightarrow \mathbf{\log_3 \frac{1}{9} = -2}$

(4) $5^{\frac{1}{2}} = \sqrt{5} \Rightarrow \mathbf{\log_5 \sqrt{5} = \frac{1}{2}}$

2. Rewrite each of the following relationships in the form $a^n = N$.

Ex.
$\log_2 8 = 3 \Rightarrow 2^3 = 8, \quad \log_2 \frac{1}{8} = -3 \Rightarrow 2^{-3} = \frac{1}{8}$

(1) $\log_3 9 = 2 \Rightarrow \mathbf{3^2 = 9}$

(2) $\log_5 1 = 0 \Rightarrow \mathbf{5^0 = 1}$

(3) $\log_2 \frac{1}{16} = -4 \Rightarrow \mathbf{2^{-4} = \frac{1}{16}}$

(4) $\log_{10} 0.001 = -3 \Rightarrow \mathbf{10^{-3} = 0.001}$

3. In each equation, find the value of x.

Ex.

$\log_5 125 = x$

[Sol] $5^x = 125 = 5^3$

$x = 3$

Comparing the exponents, $x = 3$.

(1) $\log_2 8 = x$

[Sol] $2^x = 8 = 2^3$

$\boldsymbol{x = 3}$

(2) $\log_2 64 = x$

[Sol] $2^x = 64 = 2^6$

$\boldsymbol{x = 6}$

(3) $\log_2 128 = x$

[Sol] $2^x = 128 = 2^7$

$\boldsymbol{x = 7}$

(4) $\log_2 \frac{1}{4} = x$

[Sol] $2^x = \frac{1}{4} = 2^{-2}$

$\boldsymbol{x = -2}$

(5) $\log_2 \frac{1}{32} = x$

[Sol] $2^x = \frac{1}{32} = 2^{-5}$

$\boldsymbol{x = -5}$

(6) $\log_2 \sqrt{2} = x$

[Sol] $2^x = \sqrt{2} = 2^{\frac{1}{2}}$

$\boldsymbol{x = \frac{1}{2}}$

(7) $\log_2 \frac{1}{\sqrt{32}} = x$

[Sol] $2^x = \frac{1}{\sqrt{32}} = 2^{-\frac{5}{2}}$

$\boldsymbol{x = -\frac{5}{2}}$

Note: For $\log_a N$, where $a > 0$, $a \neq 1$ and $N > 0$:

- a is called the ***base***.
- N is called the ***antilogarithm***.
- $\log_a N$ is called the ***logarithm of N*** to the base a.
- Logarithms with base 1 are undefined. For example, consider $\log_1 2$.
  If $\log_1 2 = n$, then $1^n = 2$, but clearly no value of n exists.
  Therefore a condition $a \neq 1$ is necessary.

L 2a KUMON

Logarithmic Functions

1. In each equation, find the value of x.

(1) $\log_3 27 = x$

[Sol] $3^x = 27 = 3^3$

$\boldsymbol{x = 3}$

(2) $\log_3 81 = x$

[Sol] $3^x = 81 = 3^4$

$\boldsymbol{x = 4}$

(3) $\log_5 125 = x$

[Sol] $5^x = 125 = 5^3$

$\boldsymbol{x = 3}$

(4) $\log_5 5 = x$

[Sol] $5^x = 5 = 5^1$

$\boldsymbol{x = 1}$

(5) $\log_5 1 = x$

[Sol] $5^x = 1 = 5^0$

$\boldsymbol{x = 0}$

(6) $\log_4 8 = x$

[Sol] $4^x = 8$

$2^{2x} = 2^3$

$\boldsymbol{x = \frac{3}{2}}$

(7) $\log_8 32 = x$

[Sol] $8^x = 32$

$2^{3x} = 2^5$

$\boldsymbol{x = \frac{5}{3}}$

(8) $\log_9 27 = x$

[Sol] $9^x = 27$

$3^{2x} = 3^3$

$\boldsymbol{x = \frac{3}{2}}$

L 2b

2. In each equation, find the value of x.

Ex.

$\log_x 9 = 2$

[Sol] $x^2 = 9 = 3^2$

Since $x > 0$, $x = 3$

Ex.

$\log_3 x = 2$

[Sol] $3^2 = x$

$x = 9$

(1) $\log_x 27 = 3$

[Sol] $x^3 = 27 = 3^3$

$\boldsymbol{x = 3}$

(2) $\log_x 64 = 3$

[Sol] $x^3 = 64 = 4^3$

$\boldsymbol{x = 4}$

(3) $\log_2 x = 5$

[Sol] $2^5 = x$

$\boldsymbol{x = 32}$

(4) $\log_5 x = 2$

[Sol] $5^2 = x$

$\boldsymbol{x = 25}$

(5) $\log_x \frac{1}{8} = -3$

[Sol] $x^{-3} = \frac{1}{8} = 2^{-3}$

$\boldsymbol{x = 2}$

(6) $\log_3 x = -2$

[Sol] $3^{-2} = x$

$\boldsymbol{x = \frac{1}{9}}$

(7) $\log_5 1 = x$

[Sol] $5^x = 1 = 5^0$

$\boldsymbol{x = 0}$

(8) $\log_5 \frac{1}{125} = x$

[Sol] $5^x = \frac{1}{125} = 5^{-3}$

$\boldsymbol{x = -3}$

The relationship between exponents and logarithms can be expressed as shown in the figure below.

$\bigcirc^{\blacktriangle} = \square \Leftrightarrow \log_{\bigcirc}\square = \blacktriangle$

The base of the exponent (on the left) is the same as the base of the logarithm (on the right).

L 3a KUMON

Logarithmic Functions

In each equation, find the value of x.

(1) $\log_{10} 10 = x$

[Sol] $10^x = 10$

$\boldsymbol{x = 1}$

(2) $\log_{10} 1 = x$

[Sol] $10^x = 1 = 10^0$

$\boldsymbol{x = 0}$

(3) $\log_{10} \frac{1}{10} = x$

[Sol] $10^x = \frac{1}{10} = 10^{-1}$

$\boldsymbol{x = -1}$

(4) $\log_{10} 0.001 = x$

[Sol] $10^x = 0.001 = 10^{-3}$

$\boldsymbol{x = -3}$

(5) $\log_5 x = 2$

[Sol] $5^2 = x$

$\boldsymbol{x = 25}$

(6) $\log_2 x = -2$

[Sol] $2^{-2} = x$

$\boldsymbol{x = \frac{1}{4}}$

(7) $\log_x \frac{1}{125} = -3$

[Sol] $x^{-3} = \frac{1}{125} = 5^{-3}$

$\boldsymbol{x = 5}$

(8) $\log_x 0.125 = -3$

[Sol] $x^{-3} = 0.125 = \frac{1}{8} = 2^{-3}$

$\boldsymbol{x = 2}$

L3b

(9) $\log_2 2 = x$

[Sol] $2^x = 2$

$\boldsymbol{x = 1}$

(10) $\log_4 2 = x$

[Sol] $4^x = 2$

$2^{2x} = 2$

$\boldsymbol{x = \frac{1}{2}}$

(11) $\log_4 1 = x$

[Sol] $4^x = 1$

$\boldsymbol{x = 0}$

(12) $\log_4 \frac{1}{2} = x$

[Sol] $4^x = \frac{1}{2}$

$2^{2x} = 2^{-1}$

$\boldsymbol{x = -\frac{1}{2}}$

(13) $\log_x 25 = 2$

[Sol] $x^2 = 25$

Since $x > 0$,

$\boldsymbol{x = 5}$

(14) $\log_x 5 = 2$

[Sol] $x^2 = 5$

Since $x > 0$,

$\boldsymbol{x = \sqrt{5}}$

(15) $\log_9 x = \frac{1}{2}$

[Sol] $9^{\frac{1}{2}} = x$

$\boldsymbol{x = 3}$

(16) $\log_9 x = -\frac{1}{2}$

[Sol] $9^{-\frac{1}{2}} = x$

$\boldsymbol{x = \frac{1}{3}}$

L 4a KUMON

Logarithmic Functions

In each equation, find the value of x.

Ex.

$\log_{\sqrt{3}} 9 = x$

[Sol] $(\sqrt{3})^x = 9$

$3^{\frac{x}{2}} = 3^2$

$x = 4$

(1) $\log_{\sqrt{2}} 8 = x$

[Sol] $(\sqrt{2})^x = 8$

$2^{\frac{x}{2}} = 2^3$

$\boldsymbol{x = 6}$

(2) $\log_{\sqrt{3}} \dfrac{1}{3} = x$

[Sol] $(\sqrt{3})^x = \dfrac{1}{3}$

$3^{\frac{x}{2}} = 3^{-1}$

$\boldsymbol{x = -2}$

(3) $\log_{\sqrt{5}} \dfrac{1}{125} = x$

[Sol] $(\sqrt{5})^x = \dfrac{1}{125}$

$5^{\frac{x}{2}} = 5^{-3}$

$\boldsymbol{x = -6}$

(4) $\log_3 \sqrt{3} = x$

[Sol] $3^x = \sqrt{3} = 3^{\frac{1}{2}}$

$\boldsymbol{x = \dfrac{1}{2}}$

(5) $\log_3 \sqrt[3]{3} = x$

[Sol] $3^x = \sqrt[3]{3} = 3^{\frac{1}{3}}$

$\boldsymbol{x = \dfrac{1}{3}}$

(6) $\log_3 3\sqrt{3} = x$

[Sol] $3^x = 3\sqrt{3} = 3^{\frac{3}{2}}$

$\boldsymbol{x = \dfrac{3}{2}}$

(7) $\log_{\sqrt{2}} 4\sqrt{2} = x$

[Sol] $(\sqrt{2})^x = 4\sqrt{2}$

$2^{\frac{x}{2}} = 2^{\frac{5}{2}}$

$\boldsymbol{x = 5}$

L4b

(8) $\log_8 2 = x$

[Sol] $8^x = 2$

$2^{3x} = 2$

$\boldsymbol{x = \frac{1}{3}}$

(9) $\log_{0.25} 8 = x$

[Sol] $0.25^x = 8$

$2^{-2x} = 2^3$

$\boldsymbol{x = -\frac{3}{2}}$

(10) $\log_7 x = 0$

[Sol] $7^0 = x$

$\boldsymbol{x = 1}$

(11) $\log_{32} x = -\frac{2}{5}$

[Sol] $32^{-\frac{2}{5}} = x$

$\boldsymbol{x} = (2^5)^{-\frac{2}{5}} = 2^{-2}$

$\boldsymbol{= \frac{1}{4}}$

(12) $\log_x \frac{4}{25} = -2$

[Sol] $x^{-2} = \frac{4}{25} = \left(\frac{2}{5}\right)^2 = \left(\frac{5}{2}\right)^{-2}$

$\boldsymbol{x = \frac{5}{2}}$

(13) $\log_{100} x = \frac{3}{2}$

[Sol] $100^{\frac{3}{2}} = x$

$\boldsymbol{x} = 10^3$

$\boldsymbol{= 1000}$

(14) $\log_{25} x = -1.5$

[Sol] $25^{-1.5} = x$

$\boldsymbol{x} = (5^2)^{-\frac{3}{2}} = 5^{-3}$

$\boldsymbol{= \frac{1}{125}}$

(15) $\log_x 125 = \frac{3}{2}$

[Sol] $x^{\frac{3}{2}} = 125 = 5^3 = (5^2)^{\frac{3}{2}}$

$\boldsymbol{x} = 5^2$

$\boldsymbol{= 25}$

L 5a KUMON

Logarithmic Functions

When $M = a^x$, $\log_a M = x$ $\cdots$①

When $N = a^y$, $\log_a N = y$ $\cdots$②

$MN = a^x \times a^y = a^{x+y}$

Therefore,

$\log_a(MN) = \log_a(a^{x+y}) = x + y = \log_a M + \log_a N$ ☜ Substituting ① and ② into x and y.

Similarly, we can obtain properties **3.** and **4.** below.

Properties of Logarithms

When $a > 0$, $a \neq 1$, $M > 0$, $N > 0$,

1. $\log_a 1 = 0$, $\log_a a = 1$, $\log_a a^m = m$

2. $\log_a MN = \log_a M + \log_a N$

3. $\log_a \dfrac{M}{N} = \log_a M - \log_a N$ $\left(\log_a \dfrac{1}{N} = -\log_a N\right)$

4. $\log_a M^n = n \log_a M$ $\left(\log_a \sqrt[n]{M} = \dfrac{1}{n} \log_a M\right)$

1. Using the ***Properties of Logarithms*** above, write each of the following expressions in terms of $\log_a x$ and $\log_a y$.

(1) $\log_a xy = \log_a \boxed{x} + \log_a \boxed{y}$

(2) $\log_a \dfrac{x}{y} = \mathbf{\log_a x - \log_a y}$

(3) $\log_a \dfrac{1}{x} = \mathbf{-\log_a x}$

(4) $\log_a x^3 = \mathbf{3\log_a x}$

L5b

2. Write each of the following expressions in terms of $\log_a x$, $\log_a y$ and $\log_a z$.

(1) $\log_a xyz = \mathbf{\log_a x + \log_a y + \log_a z}$

(2) $\log_a x^3 y^2 = \mathbf{3\log_a x + 2\log_a y}$

(3) $\log_a \sqrt[3]{x^2} = \mathbf{\frac{2}{3}\log_a x}$

(4) $\log_a \sqrt{x^3} = \mathbf{\frac{3}{2}\log_a x}$

(5) $\log_a(\sqrt[3]{x^2} \cdot \sqrt{y^3}) = \log_a \sqrt[3]{x^2} + \log_a \sqrt{y^3} = \mathbf{\frac{2}{3}\log_a x + \frac{3}{2}\log_a y}$

(6) $\log_a \frac{xy}{z} = \mathbf{\log_a x + \log_a y - \log_a z}$

(7) $\log_a \frac{x^2 y^{-4}}{z} = \log_a x^2 + \log_a y^{-4} - \log_a z = \mathbf{2\log_a x - 4\log_a y - \log_a z}$

L 6a KUMON

Logarithmic Functions

Given that $\log_{10} 2 = a$ and $\log_{10} 3 = b$, write the following in terms of a and b.

(1) $\log_{10} 4 = \log_{10} 2^2 = 2\log_{10} 2 = \boldsymbol{2a}$

(2) $\log_{10} 5 = \log_{10} \dfrac{\boxed{\mathbf{10}}}{2} = \log_{10} 10 - \log_{10} 2 = \boldsymbol{1-a}$

(3) $\log_{10} 6 = \log_{10}(2\cdot 3) = \log_{10} 2 + \log_{10} 3 = \boldsymbol{a+b}$

(4) $\log_{10} 8 = \log_{10} 2^3 = 3\log_{10} 2 = \boldsymbol{3a}$

(5) $\log_{10} 9 = \log_{10} 3^2 = 2\log_{10} 3 = \boldsymbol{2b}$

(6) $\log_{10} 12 = \log_{10}(2^2\cdot 3) = 2\log_{10} 2 + \log_{10} 3 = \boldsymbol{2a+b}$

(7) $\log_{10} 18 = \log_{10}(2\cdot 3^2) = \log_{10} 2 + 2\log_{10} 3 = \boldsymbol{a+2b}$

L6b

(8) $\log_{10} 36 = \log_{10}(2^2 \cdot 3^2) = 2\log_{10} 2 + 2\log_{10} 3 = \boldsymbol{2a+2b}$

(9) $\log_{10} 72 = \log_{10}(2^3 \cdot 3^2) = 3\log_{10} 2 + 2\log_{10} 3 = \boldsymbol{3a+2b}$

(10) $\log_{10} \dfrac{2}{3} = \log_{10} 2 - \log_{10} 3 = \boldsymbol{a-b}$

(11) $\log_{10} \dfrac{3}{2} = \log_{10} 3 - \log_{10} 2 = \boldsymbol{b-a}$

(12) $\log_{10} \dfrac{10}{3} = \log_{10} 10 - \log_{10} 3 = \boldsymbol{1-b}$

(13) $\log_{10} \dfrac{1}{2} = \log_{10} 1 - \log_{10} 2 = 0 - \log_{10} 2 = \boldsymbol{-a}$

〔 $= \log_{10} 2^{-1} = -\log_{10} 2 = \boldsymbol{-a}$ 〕

(14) $\log_{10} \dfrac{1}{6} = \log_{10} 1 - \log_{10} 6 = -\log_{10}(2 \cdot 3) = -\log_{10} 2 - \log_{10} 3 = \boldsymbol{-a-b}$

〔 $= \log_{10} 6^{-1} = -\log_{10} 6 = -\log_{10}(2 \cdot 3) = -\log_{10} 2 - \log_{10} 3 = \boldsymbol{-a-b}$ 〕

(15) $\log_{10} 0.6 = \log_{10} \dfrac{6}{10} = \log_{10} \dfrac{2 \cdot 3}{10} = \log_{10} 2 + \log_{10} 3 - \log_{10} 10 = \boldsymbol{a+b-1}$

L 7a KUMON

Logarithmic Functions

1. Given that $\log_{10} 2 = a$ and $\log_{10} 3 = b$, write the following in terms of a and b.

(1) $\log_{10} 54 = \log_{10}(2 \cdot 3^3) = \log_{10} 2 + 3\log_{10} 3 = \boldsymbol{a+3b}$

(2) $\log_{10} \dfrac{3}{4} = \log_{10} \dfrac{3}{2^2} = \log_{10} 3 - 2\log_{10} 2 = \boldsymbol{b-2a}$

(3) $\log_{10} \dfrac{8}{9} = \log_{10} \dfrac{2^3}{3^2} = 3\log_{10} 2 - 2\log_{10} 3 = \boldsymbol{3a-2b}$

(4) $\log_{10} \dfrac{5}{6} = \log_{10} \dfrac{10}{12} = \log_{10} \dfrac{10}{2^2 \cdot 3} = 1 - 2\log_{10} 2 - \log_{10} 3 = \boldsymbol{1-2a-b}$

(5) $\log_{10} \sqrt{12} = \dfrac{1}{2}\log_{10}(2^2 \cdot 3) = \log_{10} 2 + \dfrac{1}{2}\log_{10} 3 = \boldsymbol{a+\dfrac{b}{2}}$

(6) $\log_{10} \sqrt{5} = \log_{10} \sqrt{\dfrac{10}{2}} = \dfrac{1}{2}\log_{10} \dfrac{10}{2} = \dfrac{1}{2}(1 - \log_{10} 2) = \boldsymbol{\dfrac{1}{2}(1-a)}$

L 7b

2. Evaluate the following expressions.

Ex.

$$\log_3 18+\log_3 \frac{3}{2}=\log_3\left(18\cdot\frac{3}{2}\right)=\log_3 27=\log_3 3^3=3$$

(1) $\log_6 \frac{9}{2}+\log_6 8=\log_6\left(\frac{9}{2}\cdot 8\right)=\log_6 36=\log_6 6^2=\mathbf{2}$

(2) $\log_2 \frac{4}{3}+2\log_2 \sqrt{12}=\log_2 \frac{4}{3}+\log_2 12=\log_2\left(\frac{4}{3}\cdot 12\right)=\log_2 16$

$=\log_2 2^4=\mathbf{4}$

(3) $\log_6 3-\log_6 \frac{1}{2}=\log_6 3+\log_6 2=\log_6(3\cdot 2)=\log_6 6=\mathbf{1}$

(4) $\log_5 10-\log_5 \frac{2}{\sqrt{5}}=\log_5 10+\log_5 \frac{\sqrt{5}}{2}=\log_5\left(10\cdot\frac{\sqrt{5}}{2}\right)=\log_5 5\sqrt{5}$

$=\log_5 5^{\frac{3}{2}}=\mathbf{\frac{3}{2}}$

L 8a KUMON

Logarithmic Functions

1. Evaluate the following expressions.

(1) $\log_3 54 - \log_3 2 = \log_3 \frac{54}{2} = \log_3 27 = \log_3 3^3 = \mathbf{3}$

(2) $\frac{\log_3 32}{\log_3 8} = \frac{\log_3 2^5}{\log_3 2^3} = \frac{5\log_3 2}{3\log_3 2} = \mathbf{\frac{5}{3}}$

(3) $\frac{1}{6}\log_2 25 - \frac{1}{3}\log_2 10 = \frac{1}{6}\log_2 5^2 - \frac{1}{3}(\log_2 2 + \log_2 5)$

$= \frac{1}{3}\log_2 5 - \frac{1}{3} - \frac{1}{3}\log_2 5$

$= -\mathbf{\frac{1}{3}}$

L 8b

2. Evaluate the following expressions.

Ex.

$$\log_3 54 + \log_3 6 - 2\log_3 2$$
$$= \log_3(2\cdot 3^3) + \log_3(2\cdot 3) - 2\log_3 2$$
$$= \log_3 2 + 3 + \log_3 2 + 1 - 2\log_3 2$$
$$= 4$$

(1) $\log_3 27 + \log_3 18 - \log_3 2$

$$= \log_3 3^3 + \log_3(2\cdot 3^2) - \log_3 2$$
$$= 3 + \log_3 2 + 2 - \log_3 2$$
$$= \mathbf{5}$$

Alternative Solution

$$= \log_3\left(\frac{3^3\cdot 2\cdot 3^2}{2}\right)$$
$$= \log_3(3^5) = 5$$

Similar solutions are possible for (2) and (3).

(2) $\log_4 28 - \log_4 21 + \log_4 12$

$$= \log_4(4\cdot 7) - \log_4(3\cdot 7) + \log_4(3\cdot 4)$$
$$= 1 + \log_4 7 - (\log_4 3 + \log_4 7) + \log_4 3 + 1$$
$$= \mathbf{2}$$

(3) $2\log_5 15 + \log_5 4 - 2\log_5 6$

$$= 2\log_5(3\cdot 5) + \log_5 2^2 - 2\log_5(2\cdot 3)$$
$$= 2(\log_5 3 + 1) + 2\log_5 2 - 2(\log_5 2 + \log_5 3)$$
$$= \mathbf{2}$$

L 9a KUMON

Logarithmic Functions

1. Evaluate the following expressions.

(1) $2\log_3 6-\log_3 56+\log_3 14$

$=2\log_3(2\cdot3)-\log_3(2^3\cdot7)+\log_3(2\cdot7)$

$=2(\log_3 2+1)-(3\log_3 2+\log_3 7)+\log_3 2+\log_3 7$

$=\mathbf{2}$

Alternative Solution

$=\log_3(2^2\cdot3^2)-\log_3(2^3\cdot7)+\log_3(2\cdot7)$

$=\log_3\left(\dfrac{2^2\cdot3^2\cdot2\cdot7}{2^3\cdot7}\right)=\log_3(3^2)=2$

Similar solutions are possible for (2) and (3).

(2) $3\log_5 3-2\log_5 75-\log_5 15$

$=3\log_5 3-2\log_5(3\cdot5^2)-\log_5(3\cdot5)$

$=3\log_5 3-2(\log_5 3+2)-(\log_5 3+1)$

$=\mathbf{-5}$

(3) $6\log_2\dfrac{2}{3}+2\log_2\dfrac{1}{6}-4\log_2\dfrac{2}{9}$

$=6\log_2\dfrac{2}{3}-2\log_2(2\cdot3)-4\log_2\dfrac{2}{3^2}$

$=6(1-\log_2 3)-2(1+\log_2 3)-4(1-2\log_2 3)$

$=\mathbf{0}$

L9b

2. Evaluate the following expressions.

Ex.

$$\log_{10}25+\log_{10}2-\frac{1}{2}\log_{10}25$$
$$=2\log_{10}5+\log_{10}2-\log_{10}5$$
$$=\log_{10}5+\log_{10}2$$
$$=1$$

(1) $\log_{10}28-\log_{10}21+\log_{10}75$

$=\log_{10}(2^2\cdot7)-\log_{10}(3\cdot7)+\log_{10}(3\cdot5^2)$

$=2\log_{10}2+\log_{10}7-(\log_{10}3+\log_{10}7)+\log_{10}3+2\log_{10}5$

$=2(\log_{10}2+\log_{10}5)$

$=\mathbf{2}$

Alternative Solution

$=\log_{10}\left(\frac{2^2\cdot7\cdot3\cdot5^2}{3\cdot7}\right)$

$=\log_{10}(10^2)=2$

Similar solutions are possible for (2) and (3).

(2) $\log_{10}150+2\log_{10}3-\log_{10}135$

$=\log_{10}(3\cdot5\cdot10)+2\log_{10}3-\log_{10}(3^3\cdot5)$

$=\log_{10}3+\log_{10}5+1+2\log_{10}3-(3\log_{10}3+\log_{10}5)$

$=\mathbf{1}$

(3) $2\log_{10}50-\log_{10}5+\log_{10}2$

$=2\log_{10}(2\cdot5^2)-\log_{10}5+\log_{10}2$

$=2(\log_{10}2+2\log_{10}5)-\log_{10}5+\log_{10}2$

$=3(\log_{10}2+\log_{10}5)$

$=\mathbf{3}$

L 10a KUMON

Logarithmic Functions

Evaluate the following expressions.

(1) $\log_2 72 - \log_2 36$

$= \log_2 \dfrac{72}{36}$

$= \log_2 2$

$= \mathbf{1}$

(2) $\log_2 6 + \log_2 12 - 2\log_2 3$

$= \log_2(2\cdot 3) + \log_2(2^2\cdot 3) - 2\log_2 3$

$= 1 + \log_2 3 + 2 + \log_2 3 - 2\log_2 3$

$= \mathbf{3}$

Alternative Solution

$= \log_2\left(\dfrac{2\cdot 3\cdot 2^2\cdot 3}{3^2}\right)$

$= \log_2(2^3) = 3$

Similar solutions are possible for (3), (4) and (5).

(3) $\dfrac{\log_a 12 + \log_a 27}{\log_a 18}$

$= \dfrac{\log_a(2^2\cdot 3) + \log_a 3^3}{\log_a(2\cdot 3^2)}$

$= \dfrac{2\log_a 2 + \log_a 3 + 3\log_a 3}{\log_a 2 + 2\log_a 3}$

$= \dfrac{2(\log_a 2 + 2\log_a 3)}{\log_a 2 + 2\log_a 3}$

$= \mathbf{2}$

L 10b

（4）　$2\log_{10}\dfrac{5}{3}-\log_{10}\dfrac{7}{4}+2\log_{10}3+\dfrac{1}{2}\log_{10}49$

$=2(\log_{10}5-\log_{10}3)-(\log_{10}7-\log_{10}4)+2\log_{10}3+\log_{10}7$

$=2\log_{10}5+\log_{10}4$

$=2(\log_{10}5+\log_{10}2)$

$=\mathbf{2}$

（5）　$4\log_{10}\sqrt{150}-\log_{10}54+\log_{10}24$

$=2\log_{10}(2\cdot3\cdot5^2)-\log_{10}(2\cdot3^3)+\log_{10}(2^3\cdot3)$

$=2\log_{10}2+2\log_{10}3+4\log_{10}5-(\log_{10}2+3\log_{10}3)+3\log_{10}2+\log_{10}3$

$=4(\log_{10}2+\log_{10}5)$

$=\mathbf{4}$

L IIa KUMON

Graphs of Logarithmic Functions

When $\log_a M = x$, $a^x = M$.

Taking the logarithm, to the base b, of both sides,

$$\log_b (a^x) = \log_b M$$
$$x \log_b a = \log_b M$$
$$x = \frac{\log_b M}{\log_b a}$$

Therefore, we obtain this formula:

Logarithmic Base Conversion Formula

$$\log_a M = \frac{\log_b M}{\log_b a}$$

(where $a > 0$, $b > 0$, $M > 0$ and $a \neq 1$, $b \neq 1$)

1. Given that $\log_{10} 2 = a$ and $\log_{10} 3 = b$, write the following in terms of a and b.

Ex.

$$\log_2 6 = \frac{\log_{10} 6}{\log_{10} 2} = \frac{\log_{10} 2 + \log_{10} 3}{\log_{10} 2} = \frac{a+b}{a}$$

(1) $\log_2 20 = \dfrac{\log_{10} 20}{\log_{10} 2} = \dfrac{\log_{10} 2 + 1}{\log_{10} 2} = \boldsymbol{\dfrac{a+1}{a}}$

(2) $\log_3 5 = \dfrac{\log_{10} 5}{\log_{10} 3} = \dfrac{1 - \log_{10} 2}{\log_{10} 3} = \boldsymbol{\dfrac{1-a}{b}}$

(3) $\log_2 3 \cdot \log_3 4 = \dfrac{\log_{10} 3}{\log_{10} 2} \cdot \dfrac{\log_{10} 4}{\log_{10} 3} = \dfrac{\log_{10} 3}{\log_{10} 2} \cdot \dfrac{2 \log_{10} 2}{\log_{10} 3} = \mathbf{2}$

2. Given that $\log_2 3 = a$, write the following in terms of a.

(1) $\log_2 6 = 1 + \log_2 3 = \mathbf{1} + \boldsymbol{a}$

(2) $\log_3 2 = \dfrac{\log_2 2}{\log_2 3} = \dfrac{\mathbf{1}}{\boldsymbol{a}}$

(3) $\log_4 9 = \dfrac{\log_2 9}{\log_2 4} = \dfrac{2\log_2 3}{2} = \log_2 3 = \boldsymbol{a}$

(4) $\log_3 6 = \dfrac{\log_2 6}{\log_2 3} = \dfrac{1 + \log_2 3}{\log_2 3} = \dfrac{\mathbf{1} + \boldsymbol{a}}{\boldsymbol{a}}$

(5) $\log_6 3 = \dfrac{\log_2 3}{\log_2 6} = \dfrac{\log_2 3}{1 + \log_2 3} = \dfrac{\boldsymbol{a}}{\mathbf{1} + \boldsymbol{a}}$

(6) $\log_2 1.5 = \log_2 \dfrac{3}{2} = \log_2 3 - 1 = \boldsymbol{a} - \mathbf{1}$

From (4) and (5) above, we can see that $\log_a b = \dfrac{1}{\log_b a}$.

L 12a KUMON

Graphs of Logarithmic Functions

1. Given that $\log_{10} 2 = a$ and $\log_{10} 3 = b$, write the following in terms of a and b.

(1) $\log_{10} 5 = 1 - \log_{10} 2 = \boldsymbol{1-a}$

(2) $\log_{10} 12 = 2\log_{10} 2 + \log_{10} 3 = \boldsymbol{2a+b}$

(3) $\log_2 3 = \dfrac{\log_{10} 3}{\log_{10} 2} = \dfrac{\boldsymbol{b}}{\boldsymbol{a}}$

(4) $\log_2 5 = \dfrac{\log_{10} 5}{\log_{10} 2} = \dfrac{1-\log_{10} 2}{\log_{10} 2} = \dfrac{\boldsymbol{1-a}}{\boldsymbol{a}}$

(5) $\log_6 18 = \dfrac{\log_{10} 18}{\log_{10} 6} = \dfrac{\log_{10} 2 + 2\log_{10} 3}{\log_{10} 2 + \log_{10} 3} = \dfrac{\boldsymbol{a+2b}}{\boldsymbol{a+b}}$

(6) $\log_{18} \dfrac{8}{9} = \dfrac{\log_{10}\left(\dfrac{8}{9}\right)}{\log_{10} 18} = \dfrac{3\log_{10} 2 - 2\log_{10} 3}{\log_{10} 2 + 2\log_{10} 3} = \dfrac{\boldsymbol{3a-2b}}{\boldsymbol{a+2b}}$

L 12b

2. Given that $\log_2 3 = a$ and $\log_3 7 = b$, write the following in terms of a and b.

Ex.

$$\log_6 21 = \frac{\log_3 21}{\log_3 6} = \frac{1+\log_3 7}{\log_3 2+1} = \frac{1+\log_3 7}{\dfrac{1}{\log_2 3}+1} = \frac{1+b}{\dfrac{1}{a}+1} = \frac{a+ab}{1+a}$$

(1) $\log_7 2 = \dfrac{\log_3 2}{\log_3 7} = \dfrac{\dfrac{1}{\log_2 3}}{\log_3 7} = \dfrac{1}{\log_2 3 \cdot \log_3 7} = \boldsymbol{\dfrac{1}{ab}}$

(2) $\log_{14} 56 = \dfrac{\log_3 56}{\log_3 14} = \dfrac{3\log_3 2+\log_3 7}{\log_3 2+\log_3 7} = \dfrac{\dfrac{3}{a}+b}{\dfrac{1}{a}+b} = \boldsymbol{\dfrac{3+ab}{1+ab}}$

(3) $\log_{42} 28 = \dfrac{\log_3 28}{\log_3 42} = \dfrac{2\log_3 2+\log_3 7}{\log_3 2+1+\log_3 7} = \dfrac{\dfrac{2}{a}+b}{\dfrac{1}{a}+1+b}$

$= \boldsymbol{\dfrac{2+ab}{1+a+ab}}$

L 13a KUMON

Graphs of Logarithmic Functions

1. Using the ***Logarithmic Base Conversion Formula***, evaluate the following expressions.

Ex.

$$\log_2 3 \cdot \log_3 8 = \log_2 3 \cdot \frac{\log_2 8}{\log_2 3} = 3$$

(1) $\log_2 3 \cdot \log_3 2 = \log_2 3 \cdot \dfrac{1}{\log_2 3} = \mathbf{1}$

(2) $\log_4 3 \cdot \log_9 32 = \dfrac{\log_2 3}{\log_2 4} \cdot \dfrac{\log_2 32}{\log_2 9} = \dfrac{\log_2 3}{2} \cdot \dfrac{5}{2\log_2 3} = \mathbf{\dfrac{5}{4}}$

(3) $\log_3 5 \cdot \log_5 7 \cdot \log_7 9 = \log_3 5 \cdot \dfrac{\log_3 7}{\log_3 5} \cdot \dfrac{\log_3 9}{\log_3 7} = \mathbf{2}$

(4) $\log_2 6 - \log_4 9 = \log_2 6 - \dfrac{\log_2 9}{\log_2 4} = 1 + \log_2 3 - \dfrac{2\log_2 3}{2} = \mathbf{1}$

(5) $(\log_5 2 + \log_5 3 \cdot \log_3 4)\log_2 5$

$$= \left(\frac{1}{\log_2 5} + \frac{\log_2 3}{\log_2 5} \cdot \frac{\log_2 4}{\log_2 3}\right)\log_2 5 = \frac{3}{\log_2 5} \cdot \log_2 5 = \mathbf{3}$$

From (1), we can derive the following formula: $\log_a b \cdot \log_b a = 1$.

2. Evaluate $(\log_2 3+\log_4 9)(\log_3 2+\log_9 4)$ using the following methods:

(1) Convert all the logarithms to base 2.

[Sol] $(\log_2 3+\log_4 9)(\log_3 2+\log_9 4)$

$$=\left(\log_2 3+\frac{\log_2 9}{\log_2 4}\right)\left(\frac{\log_2 2}{\log_2 3}+\frac{\log_2 4}{\log_2 9}\right)$$

$$=\left(\log_2 3+\frac{2\log_2 3}{2}\right)\left(\frac{1}{\log_2 3}+\frac{2}{2\log_2 3}\right)$$

$$=2\log_2 3\cdot\frac{2}{\log_2 3}$$

$$=\mathbf{4}$$

(2) Convert all the logarithms to base 3.

[Sol] $(\log_2 3+\log_4 9)(\log_3 2+\log_9 4)$

$$=\left(\frac{\log_3 3}{\log_3 2}+\frac{\log_3 9}{\log_3 4}\right)\left(\log_3 2+\frac{\log_3 4}{\log_3 9}\right)$$

$$=\left(\frac{1}{\log_3 2}+\frac{2}{2\log_3 2}\right)\left(\log_3 2+\frac{2\log_3 2}{2}\right)$$

$$=\frac{2}{\log_3 2}\cdot 2\log_3 2$$

$$=\mathbf{4}$$

From (1) and (2), we get the same result regardless of which base number we use.

L 14a KUMON

Graphs of Logarithmic Functions

Evaluate the following expressions.

(1) $\log_3 36 - 4\log_9 30 + 16\log_{81}\sqrt{15}$

$= \log_3 36 - \dfrac{4\log_3 30}{\log_3 9} + \dfrac{16\log_3\sqrt{15}}{\log_3 81}$

$= 2(\log_3 2 + 1) - \dfrac{4(\log_3 2 + 1 + \log_3 5)}{2} + \dfrac{8(1 + \log_3 5)}{4}$

$= 2(\log_3 2 + 1) - 2(\log_3 2 + 1 + \log_3 5) + 2(1 + \log_3 5)$

$= \mathbf{2}$

(2) $\log_2 6 \cdot \log_3 6 - (\log_2 3 + \log_3 2)$

$= (1 + \log_2 3)(\log_3 2 + 1) - (\log_2 3 + \log_3 2)$

$= (\log_3 2 + 1 + \log_2 3 \cdot \log_3 2 + \log_2 3) - (\log_2 3 + \log_3 2)$

$= 1 + \log_2 3 \cdot \log_3 2$

$= \mathbf{2}$

(3) $\log_2 14 \cdot \log_7 14 - (\log_2 7 + \log_7 2)$

$= (1 + \log_2 7)(\log_7 2 + 1) - (\log_2 7 + \log_7 2)$

$= (\log_7 2 + 1 + \log_2 7 \cdot \log_7 2 + \log_2 7) - (\log_2 7 + \log_7 2)$

$= 1 + \log_2 7 \cdot \log_7 2$

$= \mathbf{2}$

L 14b

(4) $\log_3 75 \cdot \log_5 45 - (\log_3 25 + \log_5 9)$

$= (1 + 2\log_3 5)(2\log_5 3 + 1) - (2\log_3 5 + 2\log_5 3)$

$= (2\log_5 3 + 1 + 4\log_3 5 \cdot \log_5 3 + 2\log_3 5) - (2\log_3 5 + 2\log_5 3)$

$= 1 + 4\log_3 5 \cdot \log_5 3$

$= \mathbf{5}$

Let's think about this!

Using the Definition of a Logarithm (i.e. when $a^n = N$, $n = \log_a N$), simplify the following expressions.

(i) $a^{\log_a b} = \boxed{\boldsymbol{b}}$

Let $\log_a b = n$.
Thus $a^n = b$, and
$a^{\log_a b} = a^n = b$

(iv) $2^{-\log_2 a} = \boldsymbol{\frac{1}{a}}$

(ii) $2^{\log_2 \frac{1}{2}} = \boldsymbol{\frac{1}{2}}$

(v) $a^{a\log_a 2} = \boldsymbol{2^a}$

(iii) $a^{2\log_a b} = \boldsymbol{b^2}$

(vi) $3^{\log_9 4} = 3^{\frac{\log_3 4}{\log_3 9}} = 3^{\log_3 2}$

$= \mathbf{2}$

Answers: (i) b, (ii) $\frac{1}{2}$, (iii) b^2, (iv) $\frac{1}{a}$, (v) 2^a, (vi) 2

L 15a KUMON

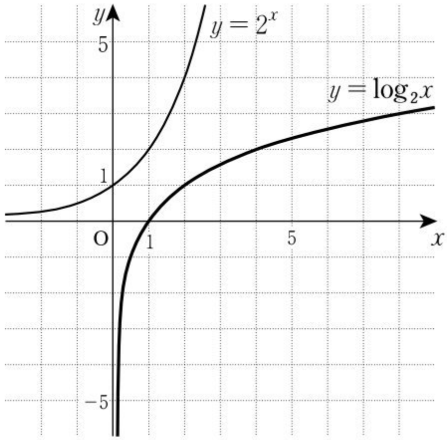
Graphs of Logarithmic Functions

Graph the following logarithmic functions.

Ex.

$y = \log_2 x$

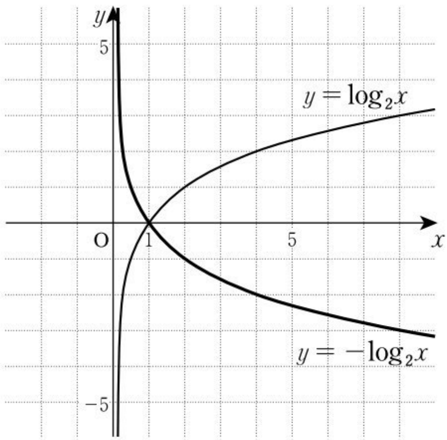
x	y
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3



Note: The asymptote of $y = \log_2 x$ is the y-axis.

(1) $y = -\log_2 x$ 〔$= \log_2 \frac{1}{x}$〕

x	y
$\frac{1}{8}$	**3**
$\frac{1}{4}$	**2**
$\frac{1}{2}$	**1**
1	**0**
2	**−1**
4	**−2**
8	**−3**

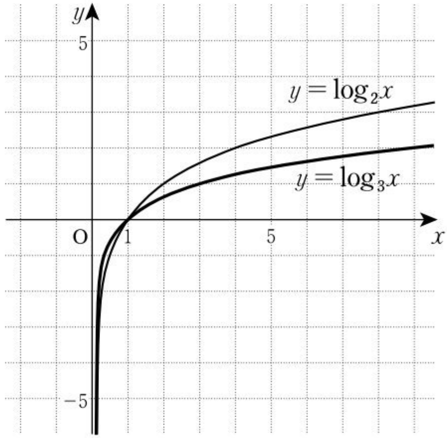


The asymptote of $y = -\log_2 x$ is the $\boxed{\textbf{y}}$ -axis.

L 15b

(2) $y = \log_3 x$

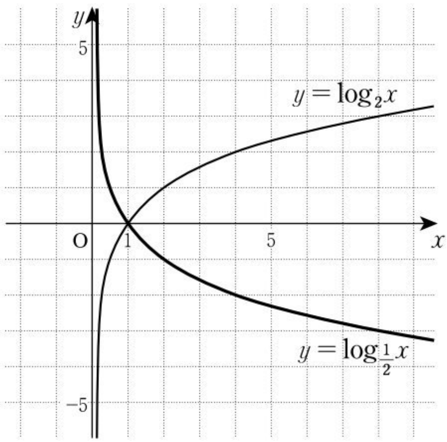
x	y
$\frac{1}{9}$	$\mathbf{-2}$
$\frac{1}{3}$	$\mathbf{-1}$
1	**0**
3	**1**
9	**2**



The asymptote of $y = \log_3 x$ is the $\boxed{\boldsymbol{y}}$-axis.

(3) $y = \log_{\frac{1}{2}} x$ $\left[= \dfrac{\log_2 x}{\log_2 \frac{1}{2}} = -\log_2 x \right]$

x	y
$\frac{1}{8}$	**3**
$\frac{1}{4}$	**2**
$\frac{1}{2}$	**1**
1	**0**
2	$\mathbf{-1}$
4	$\mathbf{-2}$
8	$\mathbf{-3}$



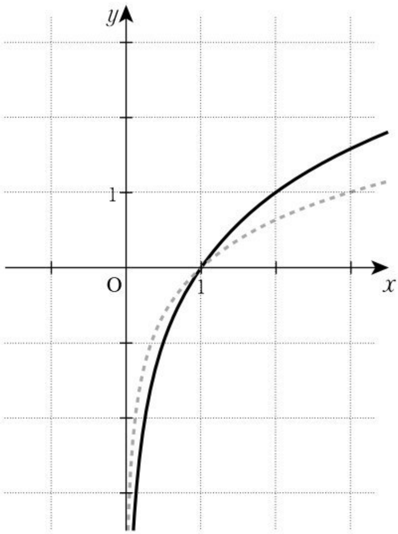
The asymptote of $y = \log_{\frac{1}{2}} x$ is the $\boxed{\boldsymbol{y}}$-axis.

L 16a KUMON

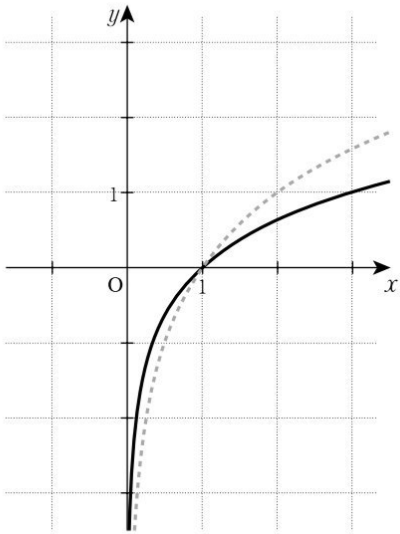
Graphs of Logarithmic Functions

1. Trace the following logarithmic functions on the grids below.

(1) $y = \log_2 x$



(2) $y = \log_3 x$



2. Complete the following using the above graphs.

• When $0 < x < 1$, $y = \log_2 x$ lies below $y = \log_3 x$.

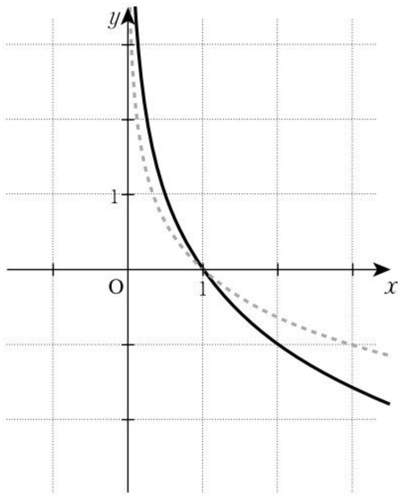
• When $x = 1$, both graphs pass through point $(1, \boxed{\mathbf{0}})$. $\log_2 1 = 0$, $\log_3 1 = 0$

• When $x > 1$, $y = \log_2 x$ lies $\boxed{\textbf{above}}$ $y = \log_3 x$.

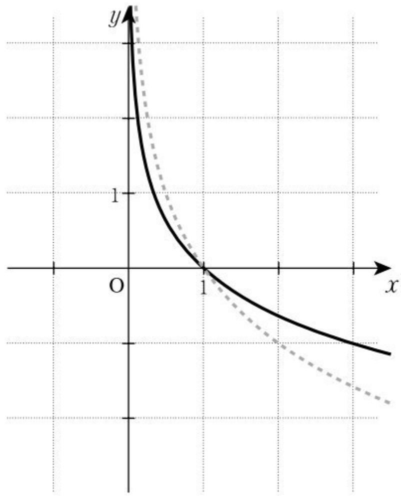
L 16b

3. Trace the following logarithmic functions on the grids below.

(1) $y=\log_{\frac{1}{2}}x$



(2) $y=\log_{\frac{1}{3}}x$



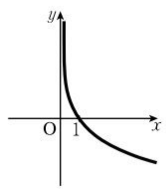
4. Complete the following using the above graphs.

- When $0<x<1$, $y=\log_{\frac{1}{2}}x$ lies $\boxed{\textbf{above}}$ $y=\log_{\frac{1}{3}}x$.
- When $x=1$, both graphs pass through point ($\boxed{\mathbf{1}}$, 0). $\log_{\frac{1}{2}}1=0$ $\log_{\frac{1}{3}}1=0$
- When $x>1$, $y=\log_{\frac{1}{2}}x$ lies $\boxed{\textbf{below}}$ $y=\log_{\frac{1}{3}}x$.

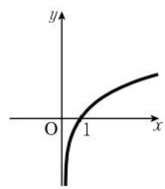
Note: When $a>0$ and $a\neq 1$, a function of the form $y=\log_a x$ is called a ***logarithmic function*** of x, where a is the base.

The graph of the logarithmic function $y=\log_a x$ passes through point $(1,\ 0)$, and its asymptote is the y-axis.

When $0<a<1$



When $a>1$

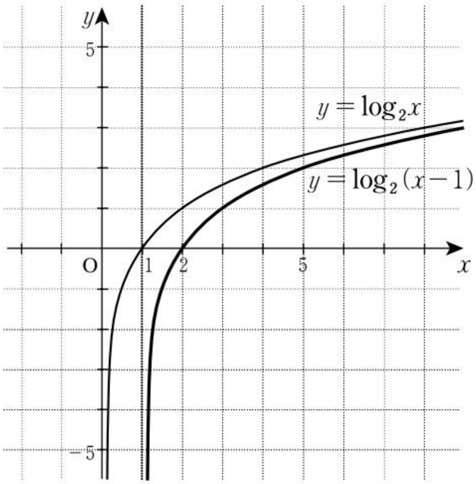


L 17a KUMON

Graphs of Logarithmic Functions

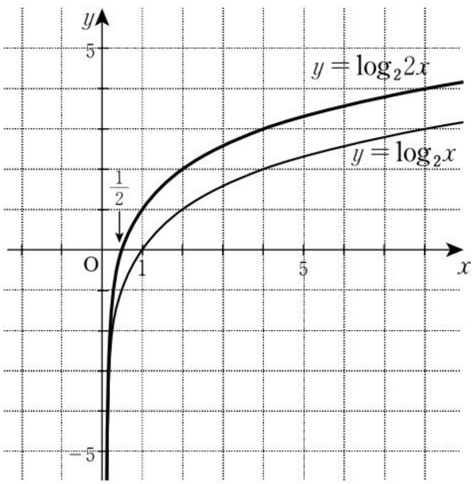
Graph the following logarithmic functions.

(1) $y=\log_2(x-1)$



$y=\log_2(x-1)$ is a translation of $y=\log_2 x$, $\boxed{1}$ unit(s) along the x-axis.

(2) $y=\log_2 2x$

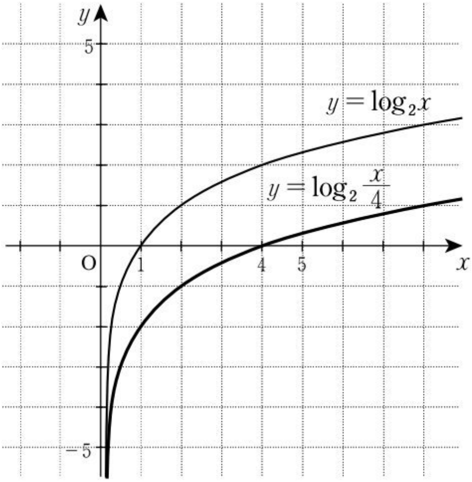


$$y=\log_2 2x$$
$$=\log_2 x+\log_2 2$$
$$=\log_2 x+1$$

$y=\log_2 2x$ is a translation of $y=\log_2 x$, $\boxed{1}$ unit(s) along the y-axis.

L 17b

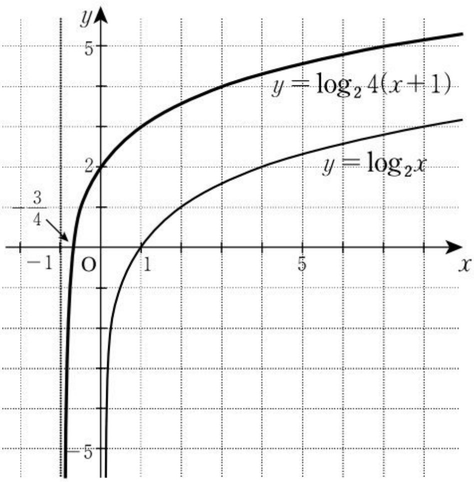
(3) $y=\log_2 \frac{x}{4}$



$$y=\log_2 \frac{x}{4}$$
$$=\log_2 x-\log_2 4$$
$$=\log_2 x-2$$

$y=\log_2 \frac{x}{4}$ is a translation of $y=\log_2 x$, $\boxed{\mathbf{-2}}$ unit(s) along the y-axis.

(4) $y=\log_2 4(x+1)$



$$y=\log_2 4(x+1)$$
$$=\log_2 (x+1)+\log_2 4$$
$$=\log_2 (x+1)+2$$

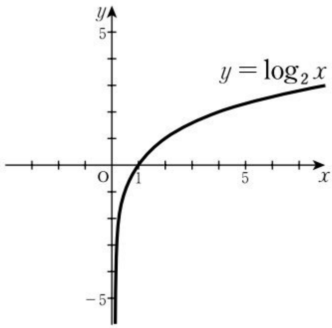
$y=\log_2 4(x+1)$ is a translation of $y=\log_2 x$, $\boxed{\mathbf{-1}}$ unit(s) along the x-axis, and $\boxed{\mathbf{2}}$ unit(s) along the y-axis.

The graph of $y=\log_a(x-p)+q$ (where $a>0$ and $a\neq 1$) is a translation of $y=\log_a x$, p units along the x-axis, and q units along the y-axis.

L 18a KUMON

Graphs of Logarithmic Functions

1. Describe how the following logarithmic functions have been translated from $y=\log_2 x$.



(1) $y=\log_2(x-3)$

[Sol] $\boxed{3}$ unit(s) along the x-axis

(2) $y=\log_2(x+4)$

[Sol] **−4 units along the *x*-axis**

(3) $y=\log_2 4x$

[Sol] $y=\log_2 4x=\log_2 x+\log_2 4=\log_2 x+2$

Ans: $\boxed{2}$ unit(s) along the y-axis

(4) $y=\log_2 \dfrac{x}{8}$

[Sol] $y=\log_2 \dfrac{x}{8}=\log_2 x-\log_2 8=\log_2 x-3$

Ans: **−3 units along the *y*-axis**

(5) $y=\log_2 2(x-3)$

[Sol] $y=\log_2 2(x-3)=\log_2(x-3)+\log_2 2=\log_2(x-3)+1$

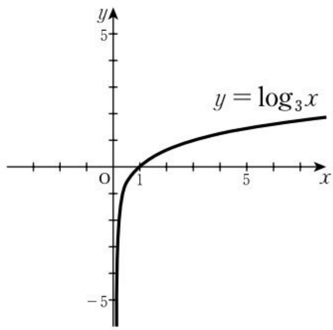
Ans: $\boxed{3}$ unit(s) along the x-axis, and $\boxed{1}$ unit(s) along the y-axis

L 18b

2. Find the function that results when the graph of the function $y = \log_3 x$ is translated in the following ways:

(1) 1 unit along the x-axis

[Sol] $\boldsymbol{y = \log_3(x-1)}$



(2) -3 units along the x-axis

[Sol] $\boldsymbol{y = \log_3(x+3)}$

(3) 1 unit along the y-axis

[Sol] $\boldsymbol{y = \log_3 x + 1}$

(4) -2 units along the y-axis

[Sol] $\boldsymbol{y = \log_3 x - 2}$

(5) 2 units along the x-axis, and -3 units along the y-axis

[Sol] $\boldsymbol{y = \log_3(x-2) - 3}$

L 19a KUMON

Graphs of Logarithmic Functions

1. Order the following logarithms.

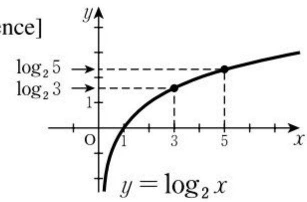
Ex.

$\log_2 3,\quad \log_2 5$

[Sol] The base is >1, and $3<5$.
Therefore, $\log_2 3<\log_2 5$

Since the base is greater than 1, the logarithms take the same order of the antilogarithms.

[Reference]



(1) $\log_3 2,\quad \log_3 5$

[Sol] The base is >1, and $2<5$.
Therefore, $\mathbf{\log_3 2<\log_3 5}$

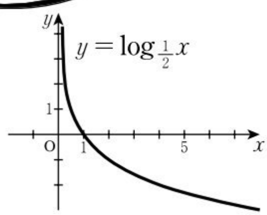
(2) $2\log_2 3,\quad \log_2 7$

[Sol] $2\log_2 3=\log_2 9$
The base is >1, and $7<9$.
Therefore, $\log_2 7<\log_2 9$
Thus, $\mathbf{\log_2 7<2\log_2 3}$

Since the base is less than 1, the logarithms take the reverse order of the antilogarithms.

(3) $\log_{\frac{1}{2}} 3,\quad \log_{\frac{1}{2}} 5$

[Sol] The base is <1, and $3<5$.
Therefore, $\mathbf{\log_{\frac{1}{2}} 5<\log_{\frac{1}{2}} 3}$



L 19b

2. Order the following logarithms.

Ex.

$\log_4 7,\quad \log_2 3$

[Sol] $\log_4 7 = \dfrac{\log_2 7}{\log_2 4} = \dfrac{1}{2}\log_2 7 = \log_2 7^{\frac{1}{2}}$ ☜ First, convert both logarithms to base 2.

The base is >1, and $7^{\frac{1}{2}} < 3$.

Therefore, $\log_2 7^{\frac{1}{2}} < \log_2 3$

Thus, $\log_4 7 < \log_2 3$

(1) $2\log_2 3,\quad 3\log_4 3$

[Sol] $2\log_2 3 = \log_2 3^2,\quad 3\log_4 3 = \dfrac{3\log_2 3}{\log_2 4} = \dfrac{3}{2}\log_2 3 = \log_2 3^{\frac{3}{2}}$

The base is >1, and $3^{\frac{3}{2}} < 3^2$.

Therefore, $\log_2 3^{\frac{3}{2}} < \log_2 3^2$

Thus, $\mathbf{3\log_4 3 < 2\log_2 3}$

(2) $\log_{\frac{1}{2}} 5,\quad \log_2 3$

[Sol] $\log_{\frac{1}{2}} 5 = \dfrac{\log_2 5}{\log_2 \frac{1}{2}} = -\log_2 5 = \log_2 \dfrac{1}{5}$

The base is >1, and $\dfrac{1}{5} < 3$.

Therefore, $\log_2 \dfrac{1}{5} < \log_2 3$

Thus, $\mathbf{\log_{\frac{1}{2}} 5 < \log_2 3}$

Given the logarithmic functions $y = \log_a p$ and $y = \log_a q$:

When $a > 1$, then $p < q \iff \log_a p < \log_a q$.

When $0 < a < 1$, then $p < q \iff \log_a p > \log_a q$.

The symbol $\Leftrightarrow$ means "if and only if".

L 20a

Graphs of Logarithmic Functions

1. For each function, write the letter (A)$\sim$(F) of the corresponding graph.

(1) $y=\log_2 x$ $\cdots$ $\boxed{\textbf{(B)}}$

(2) $y=\log_{\frac{1}{2}} x$ $\cdots$ $\boxed{\textbf{(E)}}$

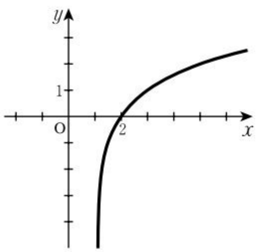
(3) $y=\log_2 2x$ $\cdots$ $\boxed{\textbf{(C)}}$

(4) $y=\log_2 \dfrac{x}{4}$ $\cdots$ $\boxed{\textbf{(F)}}$

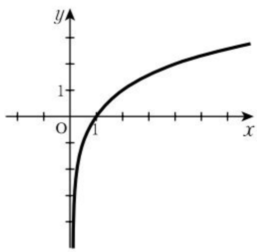
(5) $y=\log_2(x-1)$ $\cdots$ $\boxed{\textbf{(A)}}$

(6) $y=\log_2 x-1$ $\cdots$ $\boxed{\textbf{(D)}}$

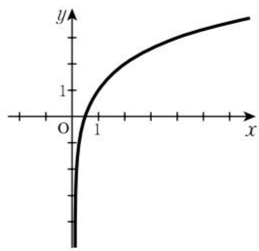
(A)



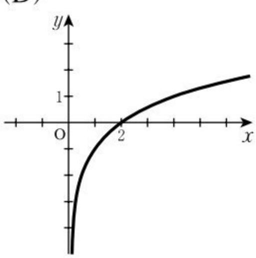
(B)



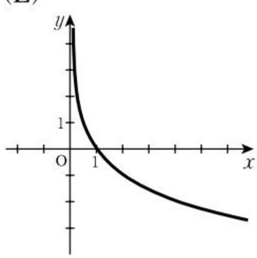
(C)



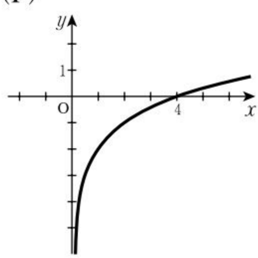
(D)



(E)



(F)



2. Describe how the following functions have been translated from $y=\log_3 x$.

(1) $y=\log_3 3x$

[Sol] $y=\log_3 x+\log_3 3=\log_3 x+1$

Therefore, **1 unit along the $\boldsymbol{y}$-axis**.

(2) $y=\log_3 \dfrac{x-3}{9}$

[Sol] $y=\log_3(x-3)-\log_3 9=\log_3(x-3)-2$

Therefore, **3 units along the $\boldsymbol{x}$-axis, and -2 units along the $\boldsymbol{y}$-axis**.

3. Order the following logarithms.

$\log_9 12,\quad 2\log_3 2$

[Sol] $\log_9 12=\dfrac{\log_3 12}{\log_3 9}=\dfrac{1}{2}\log_3 12=\log_3 12^{\frac{1}{2}},\quad 2\log_3 2=\log_3 4$

The base is >1, and $12^{\frac{1}{2}}<4$.

Therefore, $\log_3 12^{\frac{1}{2}}<\log_3 4$

Thus, $\boldsymbol{\log_9 12<2\log_3 2}$

Let's try this!

Determine which, if any, of the following logarithmic functions have the same graph.

(i) $y=\log_2 x$ (ii) $y=\log_{\frac{1}{2}} x$ (iii) $y=\log_2 \dfrac{1}{x}$

(iv) $y=-\log_2 x$ (v) $y=\log_2 2x$ (vi) $y=\log_2 x+1$

Answer: Functions (ii), (iii) and (iv) have the same graph.
Functions (v) and (vi) have the same graph.

L 21 a KUMON

Logarithmic Equations and Inequalities

1. Solve the following logarithmic equations.

Ex.

$\log_3(x-1)=2$

[Sol] $x-1=9$

$x=10$

Check:
LHS $=\log_3(10-1)=\log_3 9$
$=2=$ RHS

(1) $\log_2(5-3x)=1$

[Sol] $5-3x=2$

$-3x=-3$

$\boldsymbol{x=1}$

(2) $\log_4(x-1)=\dfrac{1}{2}$

[Sol] $x-1=2$

$\boldsymbol{x=3}$

(3) $\log_2(x+1)=-1$

[Sol] $x+1=\dfrac{1}{2}$

$\boldsymbol{x=-\dfrac{1}{2}}$

(4) $\log_{\frac{1}{2}}(2x-3)=1$

[Sol] $2x-3=\dfrac{1}{2}$

$2x=\dfrac{7}{2}$

$\boldsymbol{x=\dfrac{7}{4}}$

(5) $\log_{10}\left(\dfrac{x+3}{2}\right)=0$

[Sol] $\dfrac{x+3}{2}=1$

$x+3=2$

$\boldsymbol{x=-1}$

All the values of x found in the above questions give valid results when substituted into the original equation (i.e. the antilogarithm is >0).
In the example, the antilogarithm $=x-1=10-1=9>0$.

2. Solve the following logarithmic equations.

Ex.

$\log_2 x + \log_2(x-1) = 1$

[Sol] $\log_2 x(x-1) = 1$

$x(x-1) = 2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2,\ -1$

Since antilogarithms are >0,

$x>0,\ \ x-1>0$

So we must have $x>1$.

The solution $x=-1$ does not satisfy this condition.

Therefore, $x=2$

Check:
If $x=2$,
LHS $= \log_2 2 + \log_2(2-1)$
$= 1+0 = 1 =$ RHS

If $x=-1$,
LHS $= \log_2(-1) + \log_2(-1-1)$: extraneous solution

(1) $\log_{10} x + \log_{10}(x+3) = 1$

[Sol] $\log_{10} x(x+3) = 1$

$x(x+3) = 10$

$x^2 + 3x - 10 = 0$

$(x+5)(x-2) = 0$

$x = -5,\ 2$

Since antilogarithms are >0,

$x>0,\ \ x+3>0$

So we must have $x>0$.

The solution $x=-5$ does not satisfy this condition.

Therefore, $\boldsymbol{x=2}$

L 22a KUMON

Logarithmic Equations and Inequalities

Solve the following logarithmic equations.

(1) $\log_2(x-1)+\log_2(x+2)=2$

[Sol] $\log_2(x-1)(x+2)=2$

$(x-1)(x+2)=4$

$x^2+x-6=0$

$(x+3)(x-2)=0$

$x=-3,\ 2$

Since antilogarithms are >0,

$x-1>0,\ \ x+2>0$

So we must have $x>1$.

The solution $x=-3$ does not satisfy this condition.

Therefore, $\boldsymbol{x=2}$

(2) $2\log_{10}(x-1)=\log_{10}(x+1)$

[Sol] $\log_{10}(x-1)^2=\log_{10}(x+1)$

$(x-1)^2=x+1$

$x^2-3x=0$

$x(x-3)=0$

$x=0,\ 3$

Since antilogarithms are >0,

$x-1>0,\ \ x+1>0$

So we must have $x>1$.

The solution $x=0$ does not satisfy this condition.

Therefore, $\boldsymbol{x=3}$

L 22b

(3) $\log_5(x+1)+\log_5(x-3)=1$

[Sol] $\log_5(x+1)(x-3)=1$

$(x+1)(x-3)=5$

$x^2-2x-8=0$

$(x-4)(x+2)=0$

$x=4,\ -2$

Since antilogarithms are >0,

$x+1>0,\ \ x-3>0$

So we must have $x>3$.

The solution $x=-2$ does not satisfy this condition.

Therefore, $\boldsymbol{x=4}$

(4) $\log_2(x+1)=\log_2(2-x)+1$

[Sol] $\log_2(x+1)-\log_2(2-x)=1$

$\log_2\dfrac{x+1}{2-x}=1$

$\dfrac{x+1}{2-x}=2$

$x+1=4-2x$

$x=1$

Since antilogarithms are >0,

$x+1>0,\ \ 2-x>0$

So we must have $-1<x<2$.

The solution $x=1$ satisfies this condition.

Therefore, $\boldsymbol{x=1}$

L 23a KUMON

Logarithmic Equations and Inequalities

Solve the following logarithmic equations.

Ex.

$(1+\log_2 x)\cdot\log_2 x = 2$

[Sol] Let $\log_2 x = X$

$(1+X)X = 2$

$X^2+X-2=0$

$(X+2)(X-1)=0$

$X=-2,\ 1$

When $X=-2$,

$\log_2 x = -2$

$x = \frac{1}{4}$

When $X=1$,

$\log_2 x = 1$

$x = 2$

(1) $(\log_3 x)^2 - 5\log_3 x + 6 = 0$

[Sol] Let $\log_3 x = X$

$X^2-5X+6=0$

$(X-3)(X-2)=0$

$X=3,\ 2$

When $X=3$,

$\log_3 x = 3$

$\boldsymbol{x = 27}$

When $X=2$,

$\log_3 x = 2$

$\boldsymbol{x = 9}$

L 23b

(2) $(\log_2 x)^2 = \log_2 x^2 + 3$

[Sol] Let $\log_2 x = X$

$X^2 = 2X + 3$

$X^2 - 2X - 3 = 0$

$(X-3)(X+1) = 0$

$X = 3, -1$

When $X = 3$,

$\log_2 x = 3$

$\boldsymbol{x = 8}$

When $X = -1$,

$\log_2 x = -1$

$\boldsymbol{x = \frac{1}{2}}$

(3) $2\log_2 x - 3\log_x 2 + 5 = 0$

[Sol] Let $\log_2 x = X$

Since $x \neq 1$,

then $\log_2 x = X \neq 0$

Therefore,

Remember:
$\log_a b = \dfrac{1}{\log_b a}$

$2X - \dfrac{3}{X} + 5 = 0$

$2X^2 + 5X - 3 = 0$

(where $X \neq 0$)

$(2X-1)(X+3) = 0$

$X = \dfrac{1}{2}, -3$

When $X = \dfrac{1}{2}$,

$\log_2 x = \dfrac{1}{2}$

$\boldsymbol{x = \sqrt{2}}$

When $X = -3$,

$\log_2 x = -3$

$\boldsymbol{x = \frac{1}{8}}$

L 24a KUMON

Logarithmic Equations and Inequalities

1. Solve the following logarithmic inequalities.

Ex.

$\log_3(x-1)<2$

[Sol] Since the base is >1,

$x-1<9$

$x<10 \quad \cdots ①$

Since the antilogarithm is >0,

$x-1>0$

$x>1 \quad \cdots ②$

From ① and ②,

$1<x<10$

(1) $\log_2(x-2)<1$

[Sol] Since the base is >1,

$x-2<2$

$x<4 \quad \cdots ①$

Since the antilogarithm is >0,

$x-2>0$

$x>2 \quad \cdots ②$

From ① and ②,

$\mathbf{2<x<4}$

(2) $\log_2(5-3x)<-1$

[Sol] Since the base is >1,

$5-3x<\frac{1}{2}$

$x>\frac{3}{2} \quad \cdots ①$

Since the antilogarithm is >0,

$5-3x>0$

$x<\frac{5}{3} \quad \cdots ②$

From ① and ②,

$\boldsymbol{\frac{3}{2}<x<\frac{5}{3}}$

L 24b

2. Solve the following logarithmic inequalities.

Ex.

$\log_{\frac{1}{3}}(x-1)<1$

[Sol] Since the base is <1,

$$x-1>\frac{1}{3}$$

$$x>\frac{4}{3} \quad \cdots ①$$

Since the antilogarithm is >0,

$$x-1>0$$

$$x>1 \quad \cdots ②$$

From ① and ②,

$$x>\frac{4}{3}$$

Alternative Solution

Change to a base >1.

$$\frac{\log_3(x-1)}{\log_3\left(\frac{1}{3}\right)}<1$$

$$\frac{\log_3(x-1)}{-1}<1$$

$$\log_3(x-1)>-1$$

$$x-1>\frac{1}{3}$$

$$x>\frac{4}{3}$$

(1) $\log_{\frac{1}{2}}(2x+3)>0$

[Sol] Since the base is <1,

$$2x+3<1$$

$$x<-1 \quad \cdots ①$$

Since the antilogarithm is >0,

$$2x+3>0$$

$$x>-\frac{3}{2} \quad \cdots ②$$

From ① and ②,

$$\boldsymbol{-\frac{3}{2}<x<-1}$$

(2) $\log_{\frac{1}{4}}(x+2)<-2$

[Sol] Since the base is <1,

$$x+2>16$$

$$x>14 \quad \cdots ①$$

Since the antilogarithm is >0,

$$x+2>0$$

$$x>-2 \quad \cdots ②$$

From ① and ②,

$$\boldsymbol{x>14}$$

L 25a KUMON

Logarithmic Equations and Inequalities

Solve the following logarithmic inequalities.

Ex.

$\log_2 x + \log_2(x-1) < 1$

[Sol] $\log_2 x(x-1) < 1$

Since the base is > 1,

$x(x-1) < 2$

$x^2 - x - 2 < 0$

$(x+1)(x-2) < 0$

$-1 < x < 2 \quad \cdots①$

Since antilogarithms are > 0,

$x > 0, \quad x - 1 > 0$

So we must also have

$x > 1 \quad \cdots②$

From ① and ②,

$1 < x < 2$

(1) $\log_3(x-2) + \log_3(x-4) < 1$

[Sol] $\log_3(x-2)(x-4) < 1$

Since the base is > 1,

$(x-2)(x-4) < 3$

$x^2 - 6x + 5 < 0$

$(x-1)(x-5) < 0$

$1 < x < 5 \quad \cdots①$

Since antilogarithms are > 0,

$x - 2 > 0, \quad x - 4 > 0$

So we must also have

$x > 4 \quad \cdots②$

From ① and ②,

$\mathbf{4 < x < 5}$

L 25b

(2) $\log_{10} 5(x+1) > 2 - \log_{10}(2x-1)$

[Sol] $\log_{10} 5(x+1) + \log_{10}(2x-1) > 2$

$\log_{10} 5(x+1)(2x-1) > 2$

Since the base is >1,

$5(x+1)(2x-1) > 100$

$(x+1)(2x-1) > 20$

$2x^2 + x - 21 > 0$

$(2x+7)(x-3) > 0$

$x < -\dfrac{7}{2},\ x > 3 \quad \cdots①$

Since antilogarithms are >0,

$x+1>0, \quad 2x-1>0$

So we must also have

$x > \dfrac{1}{2} \quad \cdots②$

From ① and ②,

$\boldsymbol{x > 3}$

(3) $2\log_{\frac{1}{2}}(x-2) > \log_{\frac{1}{2}}(x+4)$

[Sol] $\log_{\frac{1}{2}}(x-2)^2 > \log_{\frac{1}{2}}(x+4)$

Since the base is <1,

$(x-2)^2 < x+4$

$x^2 - 5x < 0$

$x(x-5) < 0$

$0 < x < 5 \quad \cdots①$

Since antilogarithms are >0,

$x-2>0, \quad x+4>0$

So we must also have

$x > 2 \quad \cdots②$

From ① and ②,

$\boldsymbol{2 < x < 5}$

L 26a KUMON

Logarithmic Equations and Inequalities

Find the maximum and minimum values of the following logarithmic functions.

Ex.

$y=(\log_3 x)^2-2\log_3 x \quad \left(\frac{1}{3}\leqq x\leqq 9\right)$

[Sol] Let $\log_3 x = X$

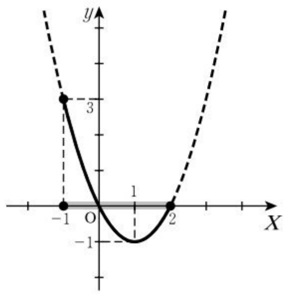
$$y=X^2-2X=(X-1)^2-1$$

Determining the range of values of X:

When $x=\frac{1}{3}$, $X=-1$

When $x=9$, $X=2$

Therefore, $-1\leqq X\leqq 2$



(Draw the graph using X on the horizontal axis.)

From the graph:

- The maximum value is 3, when $X=-1$, ☞ $y=(-1-1)^2-1=3$
  i.e. when $x=\frac{1}{3}$.
- The minimum value is -1, when $X=1$, ☞ $y=(1-1)^2-1=-1$
  i.e. when $x=3$.

(1) $y=(\log_2 x)^2-4\log_2 x+3 \quad (1\leqq x\leqq 8)$

[Sol] Let $\log_2 x = X$

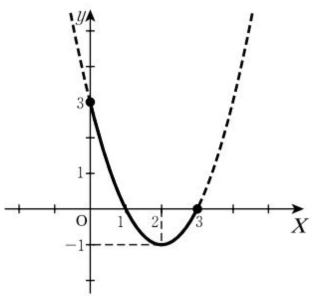
$$y=X^2-4X+3=(X-2)^2-1$$

Determining the range of values of X:

When $x=1$, $X=0$

When $x=8$, $X=3$

Therefore, $0\leqq X\leqq 3$



(Draw the graph using X on the horizontal axis.)

From the graph:

- **The maximum value is 3**, when $X=0$, i.e. **when $x=1$**.
- **The minimum value is -1**, when $X=2$, i.e. **when $x=4$**.

L 26b

(2) $y=2\log_2 x-(\log_2 x)^2 \quad (1 \leqslant x \leqslant 8)$

[Sol] Let $\log_2 x = X$

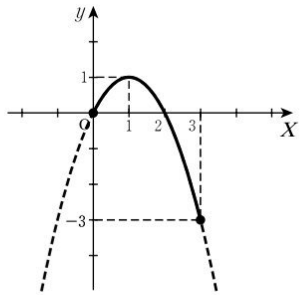
$$y=2X-X^2=-(X-1)^2+1$$

Determining the range of values of X:

When $x=1$, $X=0$

When $x=8$, $X=3$

Therefore, $0 \leqslant X \leqslant 3$



(Draw the graph using X on the horizontal axis.)

From the graph:

- **The maximum value is 1**, when $X=1$, i.e. **when $x=2$**.
- **The minimum value is -3**, when $X=3$, i.e. **when $x=8$**.

(3) $y=(\log_3 x)^2+3\log_3 x \quad (\frac{1}{3} \leqslant x \leqslant 3)$

[Sol] Let $\log_3 x = X$

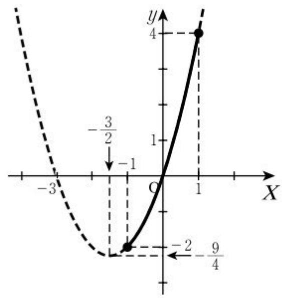
$$y=X^2+3X=\left(X+\frac{3}{2}\right)^2-\frac{9}{4}$$

Determining the range of values of X:

When $x=\frac{1}{3}$, $X=-1$

When $x=3$, $X=1$

Therefore, $-1 \leqslant X \leqslant 1$



(Draw the graph using X on the horizontal axis.)

From the graph:

- **The maximum value is 4**, when $X=1$, i.e. **when $x=3$**.
- **The minimum value is -2**, when $X=-1$, i.e. **when $x=\frac{1}{3}$**.

L 27a KUMON

Logarithmic Equations and Inequalities

When the base of a logarithm is 10, $\log_{10} N$ is called a ***common logarithm***, (or *Briggs logarithm*).
Generally, to find the value of a common logarithm, we can use the table below:

Look down the first column to find the logarithm to the tenths digit (i.e. to the first decimal place). Next, look along the top row to find the hundredths digit (i.e. the second decimal place).

Number	0	⋯	5
⋮			
2.1			.3324

$\log_{10} 2.15 = 0.3324$

1. Use the table below to find the value of the following logarithms. (Find the answers to four decimal places.)

Ex. $\log_{10} 1.05 = 0.0212$

(1) $\log_{10} 1.28 =$ **0.1072**

(2) $\log_{10} 1.8 =$ **0.2553**

(3) $\log_{10} 2.19 =$ **0.3404**

(4) $\log_{10} 2.3 =$ **0.3617**

(5) $\log_{10} 2.45 =$ **0.3892**

Table of Common Logarithms

Number	0	1	2	3	4	5	6	7	8	9
1.0	.0000	.0043	.0086	.0128	.0170	.0212	.0253	.0294	.0334	.0374
1.1	.0414	.0453	.0492	.0531	.0569	.0607	.0645	.0682	.0719	.0755
1.2	.0792	.0828	.0864	.0899	.0934	.0969	.1004	.1038	.1072	.1106
1.3	.1139	.1173	.1206	.1239	.1271	.1303	.1335	.1367	.1399	.1430
1.4	.1461	.1492	.1523	.1553	.1584	.1614	.1644	.1673	.1703	.1732
1.5	.1761	.1790	.1818	.1847	.1875	.1903	.1931	.1959	.1987	.2014
1.6	.2041	.2068	.2095	.2122	.2148	.2175	.2201	.2227	.2253	.2279
1.7	.2304	.2330	.2355	.2380	.2405	.2430	.2455	.2480	.2504	.2529
1.8	.2553	.2577	.2601	.2625	.2648	.2672	.2695	.2718	.2742	.2765
1.9	.2788	.2810	.2833	.2856	.2878	.2900	.2923	.2945	.2967	.2989
2.0	.3010	.3032	.3054	.3075	.3096	.3118	.3139	.3160	.3181	.3201
2.1	.3222	.3243	.3263	.3284	.3304	.3324	.3345	.3365	.3385	.3404
2.2	.3424	.3444	.3464	.3483	.3502	.3522	.3541	.3560	.3579	.3598
2.3	.3617	.3636	.3655	.3674	.3692	.3711	.3729	.3747	.3766	.3784
2.4	.3802	.3820	.3838	.3856	.3874	.3892	.3909	.3927	.3945	.3962

L 27b

2. Use the table on side a to find the values of the following logarithms.
(Find the answers to four decimal places.)

Ex.

$$\log_{10} 21700 = \log_{10}(2.17 \times 10^4) = \log_{10} 2.17 + \log_{10} 10^4$$
$$= 0.3365 + 4 = 4.3365$$

(1) $\log_{10} 2150$
$= \log_{10}(2.15 \times 10^3)$
$= \log_{10} 2.15 + \log_{10} 10^3$
$= 0.3324 + 3$
$= \mathbf{3.3324}$

(2) $\log_{10} 184$
$= \log_{10}(1.84 \times 10^2)$
$= \log_{10} 1.84 + \log_{10} 10^2$
$= 0.2648 + 2$
$= \mathbf{2.2648}$

(3) $\log_{10} 15.8$
$= \log_{10}(1.58 \times 10)$
$= \log_{10} 1.58 + \log_{10} 10$
$= 0.1987 + 1$
$= \mathbf{1.1987}$

(4) $\log_{10} 0.0208$
$= \log_{10}(2.08 \times 10^{-2})$
$= \log_{10} 2.08 + \log_{10} 10^{-2}$
$= 0.3181 - 2$
$= \mathbf{-1.6819}$

Let's try this!

We can find the values of logarithms that are not base 10, such as $\log_2 185$, by using the table of common logarithms.

$$\log_2 185 = \frac{\log_{10} 185}{\log_{10} \boxed{\mathbf{2}}} = \frac{\log_{10}(1.85 \times 10^2)}{\log_{10} \boxed{\mathbf{2}}} = \frac{0.2672 + 2}{0.3010} = 7.5322$$

Answers: All the answers are the same, 2.

L 28a KUMON

Logarithmic Equations and Inequalities

1. Given that $\log_{10} 2 = 0.3010$ and $\log_{10} 3 = 0.4771$, evaluate the following logarithms.

Ex.

$$\log_{10} 600 = \log_{10}(2 \cdot 3 \cdot 10^2) = \log_{10} 2 + \log_{10} 3 + 2\log_{10} 10$$
$$= 0.3010 + 0.4771 + 2 = 2.7781$$

(1) $\log_{10} 6000 = \log_{10}(2 \cdot 3 \cdot 10^3) = \log_{10} 2 + \log_{10} 3 + 3\log_{10} 10$
$= 0.3010 + 0.4771 + 3 = \mathbf{3.7781}$

(2) $\log_{10} 12000 = \log_{10}(2^2 \cdot 3 \cdot 10^3) = 2\log_{10} 2 + \log_{10} 3 + 3\log_{10} 10$
$= 2 \times 0.3010 + 0.4771 + 3 = \mathbf{4.0791}$

(3) $\log_{10} 0.3 = \log_{10}(3 \cdot 10^{-1}) = \log_{10} 3 - \log_{10} 10$
$= 0.4771 - 1 = \mathbf{-0.5229}$

(4) $\log_{10} 0.06 = \log_{10}(2 \cdot 3 \cdot 10^{-2}) = \log_{10} 2 + \log_{10} 3 - 2\log_{10} 10$
$= 0.3010 + 0.4771 - 2 = \mathbf{-1.2219}$

(5) $\log_{10} 0.0012 = \log_{10}(2^2 \cdot 3 \cdot 10^{-4}) = 2\log_{10} 2 + \log_{10} 3 - 4\log_{10} 10$
$= 2 \times 0.3010 + 0.4771 - 4 = \mathbf{-2.9209}$

L 28b

2. Given that $\log_{10} 2 = 0.3010$ and $\log_{10} 3 = 0.4771$, find the number of digits the following numbers have.

Ex.

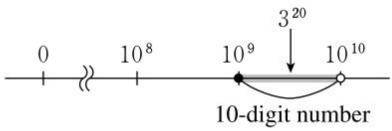
3^{20}

[Sol] $\log_{10} 3^{20} = 20\log_{10} 3 = 20 \times 0.4771 = 9.542$

$3^{20} = 10^{9.542}$

Therefore, $10^9 < 3^{20} < 10^{10}$

Thus, 3^{20} is a 10-digit number.



[Reference] $3^{20} = 3486784401$

(1) 2^{30}

[Sol] $\log_{10} 2^{30} = 30\log_{10} 2 = 30 \times 0.3010 = 9.030$

$2^{30} = 10^{9.030}$

Therefore, $10^9 < 2^{30} < 10^{10}$

Thus, 2^{30} is a **10-digit number**.

(2) 6^{20}

[Sol] $\log_{10} 6^{20} = 20\log_{10}(2\cdot 3) = 20(\log_{10} 2 + \log_{10} 3)$

$= 20(0.3010 + 0.4771) = 15.562$

$6^{20} = 10^{15.562}$

Therefore, $10^{15} < 6^{20} < 10^{16}$

Thus, 6^{20} is a **16-digit number**.

$10^0 \leqq$ (1-digit number) $< 10^1$ (1-digit numbers are $1, 2, 3, \ldots, 9$)
$10^1 \leqq$ (2-digit number) $< 10^2$ (2-digit numbers are $10, 11, 12, \ldots, 99$)
$10^2 \leqq$ (3-digit number) $< 10^3$ (3-digit numbers are $100, 101, 102, \ldots, 999$)

L 29a KUMON

Logarithmic Equations and Inequalities

1. For each given number, find the decimal place at which a non-zero digit will appear for the first time. Assume: $\log_{10} 2 = 0.3010$, $\log_{10} 3 = 0.4771$

Ex.

$\left(\frac{1}{2}\right)^{10}$

[Sol] $\log_{10}\left(\frac{1}{2}\right)^{10} = -10\log_{10} 2$

$= -3.010$

$-4 < \log_{10}\left(\frac{1}{2}\right)^{10} < -3$

Therefore, $10^{-4} < \left(\frac{1}{2}\right)^{10} < 10^{-3}$

Thus, in the number $\left(\frac{1}{2}\right)^{10}$, the first non-zero digit appears in the 4^{th} decimal place.

0.001	$(=10^{-3})$
0.0001	$(=10^{-4})$
0.00001	$(=10^{-5})$

3^{rd} decimal place, 4^{th} decimal place, 5^{th} decimal place

[Reference] $\left(\frac{1}{2}\right)^{10} = 0.0009765625$

(1) $\left(\frac{1}{6}\right)^{20}$

[Sol] $\log_{10}\left(\frac{1}{6}\right)^{20} = -20\log_{10}(2\cdot 3) = -20\times(0.3010+0.4771) = -15.562$

$-16 < \log_{10}\left(\frac{1}{6}\right)^{20} < -15$

Therefore, $10^{-16} < \left(\frac{1}{6}\right)^{20} < 10^{-15}$

Thus, in the number $\left(\frac{1}{6}\right)^{20}$, the first non-zero digit appears in the **16^{th} decimal place**.

L 29b

2. Find the minimum values of the integer n that satisfy the following inequalities. Assume: $\log_{10} 2 = 0.3010$, $\log_{10} 3 = 0.4771$

Ex.

$2^n > 10000$

[Sol] Using common logarithms,

$$\log_{10} 2^n > \log_{10} 10000$$

$$n \log_{10} 2 > 4$$

$$n > \frac{4}{\log_{10} 2} = \frac{4}{0.3010} = 13.2 \ldots$$ ☜ $\log_{10} 2 > 0$

Therefore, $n = 14$

(1) $3^n > 10^6$

[Sol] Using common logarithms,

$$\log_{10} 3^n > \log_{10} 10^6$$

$$n \log_{10} 3 > 6$$

$$n > \frac{6}{\log_{10} 3} = \frac{6}{0.4771} = 12.5 \ldots$$

Therefore, $\boldsymbol{n = 13}$

(2) $\left(\frac{1}{2}\right)^n < 0.001$

[Sol] Using common logarithms,

$$\log_{10} \left(\frac{1}{2}\right)^n < \log_{10} 10^{-3}$$

$$-n \log_{10} 2 < -3$$

$$n > \frac{3}{\log_{10} 2} = \frac{3}{0.3010} = 9.9 \ldots$$

Therefore, $\boldsymbol{n = 10}$

L 30a KUMON

Logarithmic Equations and Inequalities

1. Solve the following logarithmic equation and inequality.

(1) $\log_2(2x+3)+\log_2(x-2)=2$

[Sol] $\log_2(2x+3)(x-2)=2$

$(2x+3)(x-2)=4$

$2x^2-x-10=0$

$(2x-5)(x+2)=0$

$x=\frac{5}{2},\ -2$

Since antilogarithms are >0,

$2x+3>0,\ x-2>0$

So we must also have $x>2$.

The solution $x=-2$ does not satisfy this condition.

Therefore, $\boldsymbol{x=\frac{5}{2}}$

(2) $\log_3(x+3)+\log_3(x-5)<2$

[Sol] $\log_3(x+3)(x-5)<2$

Since the base $3>1$,

$(x+3)(x-5)<9$

$x^2-2x-24<0$

$(x+4)(x-6)<0$

$-4<x<6 \quad \cdots ①$

Since antilogarithms are >0,

$x+3>0,\ x-5>0$

So we must also have

$x>5 \quad \cdots ②$

From ① and ②,

$\boldsymbol{5<x<6}$

L 30b

2. Find the maximum and minimum values of the given logarithmic function:

$$y = -(\log_2 x)^2 + \log_2 x^4 - 1 \quad (2 \leqslant x \leqslant 16)$$

[Sol] Let $\log_2 x = X$

$$y = -X^2 + 4X - 1 = -(X-2)^2 + 3$$

Determining the range of values of X:

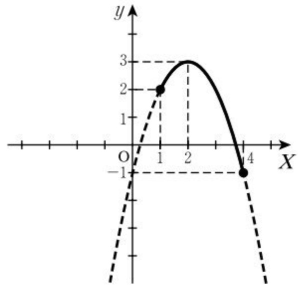
When $x = 2$, $X = 1$

When $x = 16$, $X = 4$

Therefore, $1 \leqslant X \leqslant 4$

From the graph:

- **The maximum value is 3**, when $X = 2$, i.e. **when $x = 4$**.
- **The minimum value is -1**, when $X = 4$, i.e. **when $x = 16$**.



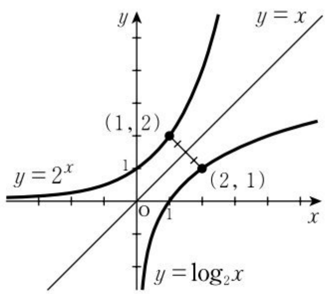
(Draw the graph using X on the horizontal axis.)

Special Note!

Study the relationship between the exponential function $y = 2^x$ and the logarithmic function $y = \log_2 x$.

Looking at the graph of $y = 2^x$ and the graph of $y = \log_2 x$, we can see that the line $y = \boxed{\boldsymbol{x}}$ is the axis of symmetry of the two functions.

Given this relationship, the exponential function $y = 2^x$ is called the ***inverse function*** of the logarithmic function $y = \log_2 x$.



Answer: x

L 31 a KUMON

Modulus Functions

The symbols $|\ \ |$ are used to denote ***absolute value***.
$|a|$ is called the ***absolute value*** of a.

When a is positive or zero, $|a| = a$
When a is negative, $|a| = -a$

1. Find the following absolute values.

Ex.
$|2| = 2$ $\qquad |0| = 0$ $\qquad |-3| = 3$

(1) $|5| = \mathbf{5}$

(2) $|-5| = \mathbf{5}$

(3) $\left|\frac{1}{2}\right| = \mathbf{\frac{1}{2}}$

(4) $\left|-\frac{1}{2}\right| = \mathbf{\frac{1}{2}}$

(5) $|-1| = \mathbf{1}$

(6) $\left|\frac{2}{3}\right| = \mathbf{\frac{2}{3}}$

(7) $|-10| = \mathbf{10}$

(8) $\left|-\frac{3}{4}\right| = \mathbf{\frac{3}{4}}$

Note: The absolute value is either positive or zero.

2. For each of the following values of x, find the values of $x-2$ and $|x-2|$.

x	-1	0	1	2	3	4	5
$x-2$	**−3**	**−2**	**−1**	**0**	**1**	**2**	**3**
$\lvert x-2\rvert$	**3**	**2**	**1**	**0**	**1**	**2**	**3**

When $x-2 \geqslant 0$, i.e. when $x \geqslant \boxed{\mathbf{2}}$, $|x-2| = x-2$

When $x-2 < 0$, i.e. when $x < \boxed{\mathbf{2}}$, $|x-2| = -(x-2)$

3. Complete the following.

(1) $|x-3|=x-3$ is true when $x \geqslant \boxed{3}$.

$|x-3|=-(x-3)$ is true when $x < \boxed{3}$.

(2) $|x+2|=x+2$ is true when $x \geqslant \boxed{-2}$.

$|x+2|=-(x+2)$ is true when $x < \boxed{-2}$.

4. Rewrite the following, removing the absolute value symbol.

Ex.

$$|x-2|=\begin{cases} x-2 & (\text{when } x \geqslant 2) \\ -(x-2) & (\text{when } x<2) \end{cases}$$

(1) $|x-1|=\begin{cases} \boldsymbol{x-1} & (\textbf{when } \boldsymbol{x \geqslant 1}) \\ \boldsymbol{-(x-1)} & (\textbf{when } \boldsymbol{x<1}) \end{cases}$

(2) $|x+3|=\begin{cases} \boldsymbol{x+3} & (\textbf{when } \boldsymbol{x \geqslant -3}) \\ \boldsymbol{-(x+3)} & (\textbf{when } \boldsymbol{x<-3}) \end{cases}$

(3) $|x|=\begin{cases} \boldsymbol{x} & (\textbf{when } \boldsymbol{x \geqslant 0}) \\ \boldsymbol{-x} & (\textbf{when } \boldsymbol{x<0}) \end{cases}$

(4) $|2x-3|=\begin{cases} \boldsymbol{2x-3} & (\textbf{when } \boldsymbol{x \geqslant \frac{3}{2}}) \\ \boldsymbol{-(2x-3)} & (\textbf{when } \boldsymbol{x<\frac{3}{2}}) \end{cases}$

L 32a

KUMON

Modulus Functions

1. For each of the following values of x, find the values of $x(x-3)$ and $|x(x-3)|$.

x	-2	-1	0	1	2	3	4	5
$x(x-3)$	**10**	**4**	**0**	**−2**	**−2**	**0**	**4**	**10**
$\|x(x-3)\|$	**10**	**4**	**0**	**2**	**2**	**0**	**4**	**10**

When $x(x-3) \geqslant 0$, i.e. when $x \leqslant \boxed{\mathbf{0}}$ or $x \geqslant \boxed{\mathbf{3}}$,

$|x(x-3)| = x(x-3)$

When $x(x-3) < 0$, i.e. when $\boxed{\mathbf{0}} < x < \boxed{\mathbf{3}}$,

$|x(x-3)| = -x(x-3)$

2. For each of the following values of x, find the values of $(x+3)(x-1)$ and $|(x+3)(x-1)|$.

x	-5	-4	-3	-2	-1	0	1	2	3
$(x+3)(x-1)$	**12**	**5**	**0**	**−3**	**−4**	**−3**	**0**	**5**	**12**
$\|(x+3)(x-1)\|$	**12**	**5**	**0**	**3**	**4**	**3**	**0**	**5**	**12**

When $(x+3)(x-1) \geqslant 0$, i.e. when $x \leqslant \boxed{\mathbf{-3}}$ or $x \geqslant \boxed{\mathbf{1}}$,

$|(x+3)(x-1)| = (x+3)(x-1)$

When $(x+3)(x-1) < 0$, i.e. when $\boxed{\mathbf{-3}} < x < \boxed{\mathbf{1}}$,

$|(x+3)(x-1)| = -(x+3)(x-1)$

L 32b

3. Rewrite the following, removing the absolute value symbol.

Ex.

$$|x(x-2)| = \begin{cases} x(x-2) & (\text{when } x \leqq 0 \text{ or } x \geqq 2) \\ -x(x-2) & (\text{when } 0 < x < 2) \end{cases}$$

(1) $|(x+1)(x-2)| = \begin{cases} (x+1)(x-2) & (\text{when } x \leqq -1 \text{ or } x \geqq 2) \\ -(x+1)(x-2) & (\text{when } -1 < x < 2) \end{cases}$

(2) $|x(x+3)| = \begin{cases} x(x+3) & (\text{when } x \leqq -3 \text{ or } x \geqq 0) \\ -x(x+3) & (\text{when } -3 < x < 0) \end{cases}$

(3) $|(x-1)(x-3)| = \begin{cases} (x-1)(x-3) & (\text{when } x \leqq 1 \text{ or } x \geqq 3) \\ -(x-1)(x-3) & (\text{when } 1 < x < 3) \end{cases}$

(4) $|x^2-9| = \begin{cases} x^2-9 & (\text{when } x \leqq -3 \text{ or } x \geqq 3) \\ -(x^2-9) & (\text{when } -3 < x < 3) \end{cases}$

(5) $|x^2-4x| = \begin{cases} x^2-4x & (\text{when } x \leqq 0 \text{ or } x \geqq 4) \\ -(x^2-4x) & (\text{when } 0 < x < 4) \end{cases}$

L 33a KUMON

Modulus Functions

Graph the following functions.

Ex.

$y=|x-1|$

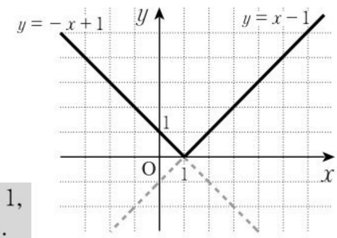
[Sol] When $x \geqslant 1$,

$y=x-1$ ☞ In the domain $x \geqslant 1$, graph $y=x-1$.

When $x<1$,

$y=-(x-1)$

$=-x+1$ ☞ In the domain $x<1$, graph $y=-x+1$.



(You do not have to draw the dashed parts of the lines.)

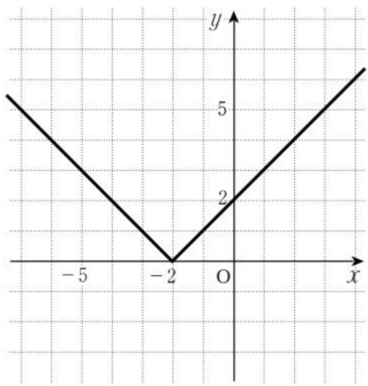
(1) $y=|x+2|$

[Sol] When $x \geqslant -2$,

$y=x+2$

When $x<-2$,

$y=-(x+2)=-x-2$



L 33b

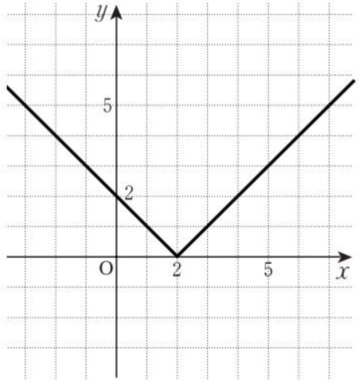
(2) $y=|x-2|$

[Sol] When $x \geqq 2$,

$y=x-2$

When $x<2$,

$y=-(x-2)=-x+2$



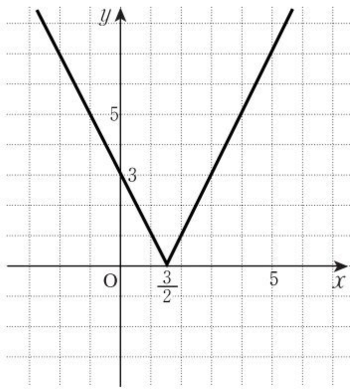
(3) $y=|2x-3|$

[Sol] When $x \geqq \dfrac{3}{2}$,

$y=2x-3$

When $x<\dfrac{3}{2}$,

$y=-(2x-3)=-2x+3$



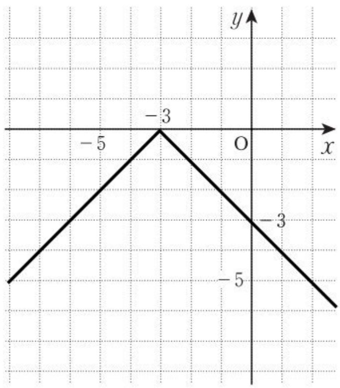
(4) $y=-|x+3|$

[Sol] When $x \geqq -3$,

$y=-(x+3)=-x-3$

When $x<-3$,

$y=x+3$



L 34a KUMON

Modulus Functions

1. Trace the graph of each function.

Ex.

$y=|x(x-2)|$

[Sol] When $x(x-2)\geqq 0$,

i.e. when $x\leqq 0$ or $x\geqq 2$,

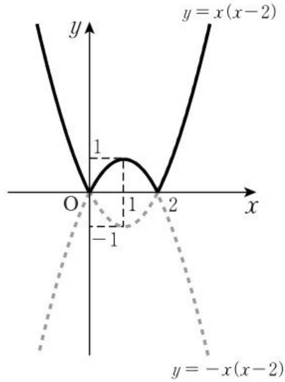
$y=x(x-2)$

When $x(x-2)<0$,

i.e. when $0<x<2$,

$y=-x(x-2)$

Note: The graph of $y=|x(x-2)|$ is the graph of $y=x(x-2)$ folded up along the x-axis.



(1) $y=|(x+1)(x-2)|$

[Sol] When $(x+1)(x-2)\geqq 0$,

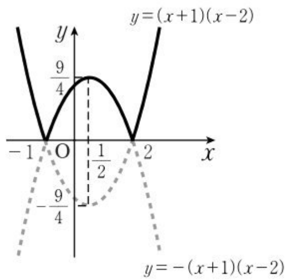
i.e. when $x\leqq -1$ or $x\geqq 2$,

$y=(x+1)(x-2)$

When $(x+1)(x-2)<0$,

i.e. when $-1<x<2$,

$y=-(x+1)(x-2)$



(2) $y=-|x^2-4|$

[Sol] When $(x^2-4)\geqq 0$,

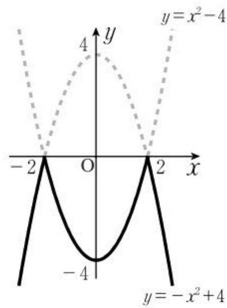
i.e. when $x\leqq -2$ or $x\geqq 2$,

$y=-x^2+4$

When $(x^2-4)<0$,

i.e. when $-2<x<2$,

$y=x^2-4$



L 34 b

2. Graph the following functions.

(1) $y=|x(x-4)|$

[Sol] When $x(x-4)\geqq 0$,

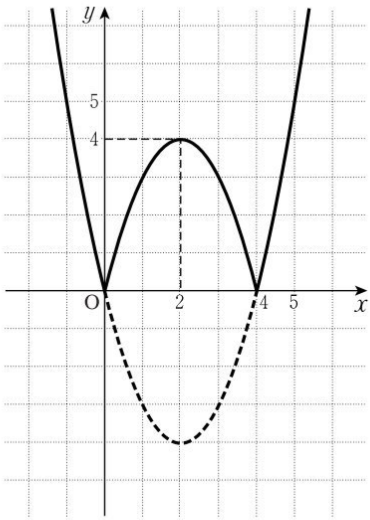
i.e. when $x\leqq 0$ or $x\geqq 4$,

$$y=x(x-4)$$
$$=(x-2)^2-4$$

When $x(x-4)<0$,

i.e. when $0<x<4$,

$$y=-x(x-4)$$
$$=-(x-2)^2+4$$



(2) $y=-|(x-1)(x-3)|$

[Sol] When $(x-1)(x-3)\geqq 0$,

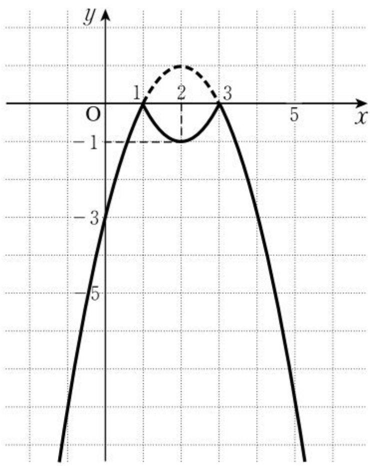
i.e. when $x\leqq 1$ or $x\geqq 3$,

$$y=-(x-1)(x-3)$$
$$=-(x-2)^2+1$$

When $(x-1)(x-3)<0$,

i.e. when $1<x<3$,

$$y=(x-1)(x-3)$$
$$=(x-2)^2-1$$



L 35a KUMON

Modulus Functions

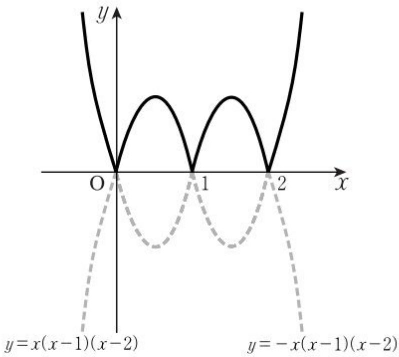
Trace the graph of each function.

Ex.

$y=|x(x-1)(x-2)|$

[Sol] When $x(x-1)(x-2)\geqslant 0$,
i.e. when $0\leqslant x\leqslant 1$ or $x\geqslant 2$,
$y=x(x-1)(x-2)$

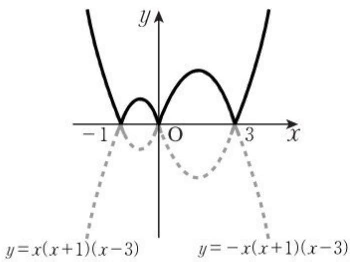
When $x(x-1)(x-2)<0$,
i.e. when $x<0$ or $1<x<2$,
$y=-x(x-1)(x-2)$



(1) $y=|x(x+1)(x-3)|$

[Sol] When $x(x+1)(x-3)\geqslant 0$,
i.e. when $-1\leqslant x\leqslant 0$ or $x\geqslant 3$,
$y=x(x+1)(x-3)$

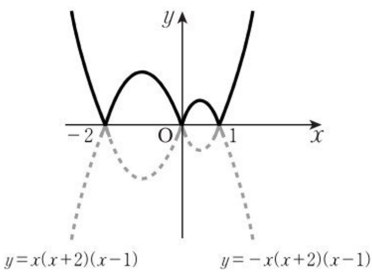
When $x(x+1)(x-3)<0$,
i.e. when $x<-1$ or $0<x<3$,
$y=-x(x+1)(x-3)$



(2) $y=|-x(x+2)(x-1)|$

[Sol] When $-x(x+2)(x-1)\geqslant 0$,
i.e. when $x\leqslant -2$ or $0\leqslant x\leqslant 1$,
$y=-x(x+2)(x-1)$

When $-x(x+2)(x-1)<0$,
i.e. when $-2<x<0$ or $x>1$,
$y=x(x+2)(x-1)$



L 35b

Ex.

$y=|x^2(x+2)|$

[Sol] When $x^2(x+2)\geqq 0$,

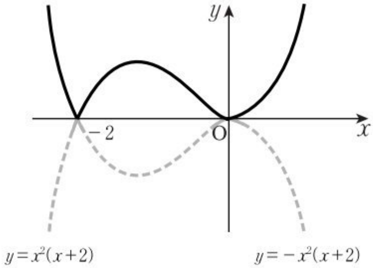
i.e. when $x\geqq -2$,

$y=x^2(x+2)$

When $x^2(x+2)<0$,

i.e. when $x<-2$,

$y=-x^2(x+2)$



(3) $y=|x(x-2)^2|$

[Sol] When $x(x-2)^2\geqq 0$,

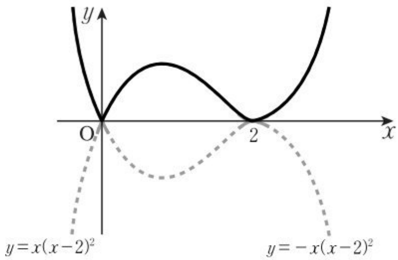
i.e. when $x\geqq 0$,

$y=x(x-2)^2$

When $x(x-2)^2<0$,

i.e. when $x<0$,

$y=-x(x-2)^2$



(4) $y=|(x-1)^2(x+2)|$

[Sol] When $(x-1)^2(x+2)\geqq 0$,

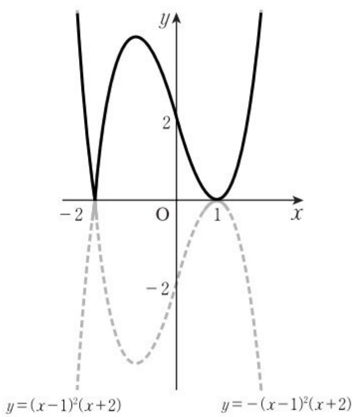
i.e. when $x\geqq -2$,

$y=(x-1)^2(x+2)$

When $(x-1)^2(x+2)<0$,

i.e. when $x<-2$,

$y=-(x-1)^2(x+2)$



L 36a KUMON

Modulus Functions

Graph the following functions.

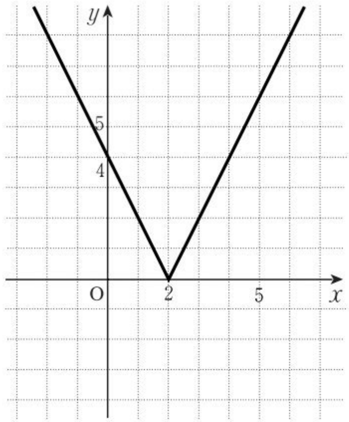
(1) $y=|2x-4|$

[Sol] When $x \geqq 2$,

$y=2x-4$

When $x<2$,

$y=-2x+4$



(2) $y=|(x+3)(x-1)|$

[Sol] When $x \leqq -3$ or $x \geqq 1$,

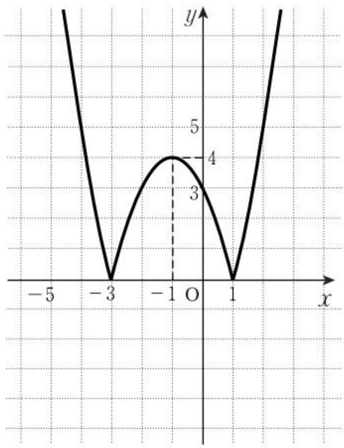
$y=(x+3)(x-1)$

$=(x+1)^2-4$

When $-3<x<1$,

$y=-(x+3)(x-1)$

$=-(x+1)^2+4$



L 36b

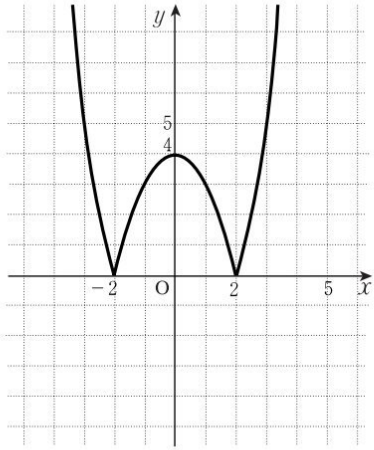
(3) $y=|x^2-4|$

[Sol] When $x \leqq -2$ or $x \geqq 2$,

$y=x^2-4$

When $-2<x<2$,

$y=-x^2+4$



(4) $y=-|x^2+2x|$

[Sol] When $x \leqq -2$ or $x \geqq 0$,

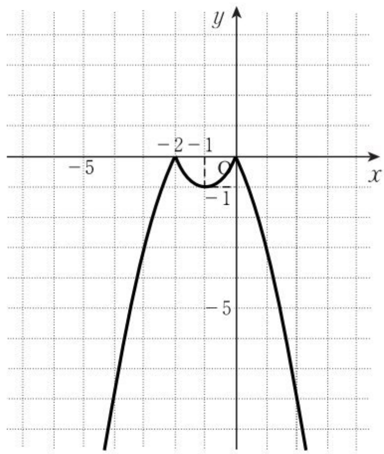
$y=-x^2-2x$

$=-(x+1)^2+1$

When $-2<x<0$,

$y=x^2+2x$

$=(x+1)^2-1$



L 37a KUMON

Modulus Functions

Find the points of intersection of the given functions.

Ex.

$y=|2x-4|$, $y=x+1$

[Sol]

(i) When $x \geqslant \boxed{2}$,

$|2x-4|=2x-4$

From $2x-4=x+1$,

First find the x-coordinate of the point of intersection.

$x=5$

$y=\boxed{5}+1=\boxed{6}$

Then find the y-coordinate of the point of intersection.

(ii) When $x<\boxed{2}$,

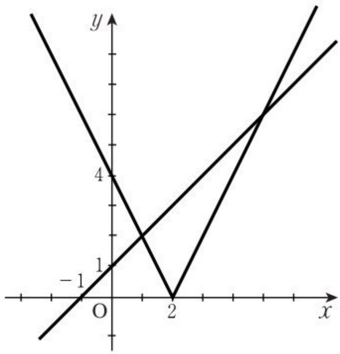
$|2x-4|=-2x+4$

From $-2x+4=x+1$,

$-3x=-3$

$x=1$

$y=\boxed{1}+1=\boxed{2}$



From (i) and (ii), the points of intersection are:

$(5, \boxed{6}), (1, \boxed{2})$

Answers: in order 2, 5, 6, 2, 1, 2, 6, 2

L 37b

(1) $y=|3x+6|$, $y=-x+2$

[Sol]

(i) When $x \geqq -2$,

$$|3x+6|=3x+6$$

From $3x+6=-x+2$,

$$4x=-4$$

$$x=-1$$

$y=-(-1)+2=3$

(ii) When $x<-2$,

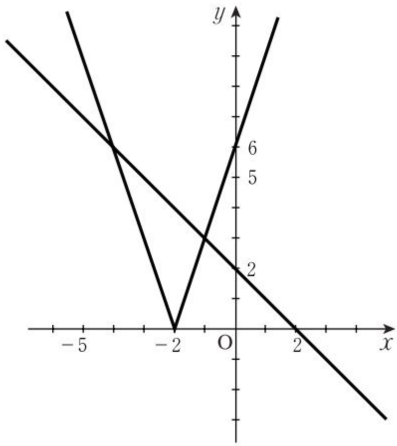
$$|3x+6|=-3x-6$$

From $-3x-6=-x+2$,

$$-2x=8$$

$$x=-4$$

$y=-(-4)+2=6$



From (i) and (ii), the points of intersection are:

$(-1,\ 3),\ (-4,\ 6)$

Modulus Functions

Find the points of intersection of the given functions.

Ex.

$y=|x^2-9|$, $y=-x+7$

[Sol]

$x^2-9=(x+3)(x-3)$

(i) When $x\leqslant -3$ or $x\geqslant 3$,

$|x^2-9|=\boxed{\boldsymbol{x^2-9}}$

From $\boxed{\boldsymbol{x^2-9}}=-x+7$,

$x^2+x-16=0$, and so $x=\dfrac{-1\pm\sqrt{65}}{2}$

When $x=\dfrac{-1+\sqrt{65}}{2}$, $y=\dfrac{15-\sqrt{65}}{2}$

When $x=\dfrac{-1-\sqrt{65}}{2}$, $y=\dfrac{15+\sqrt{65}}{2}$

(ii) When $-3<x<3$,

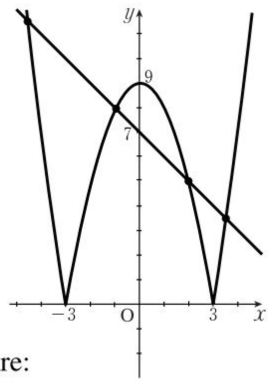
$|x^2-9|=\boxed{\boldsymbol{-x^2+9}}$

From $\boxed{\boldsymbol{-x^2+9}}=-x+7$,

$x^2-x-2=0$, and so $x=2, -1$

When $x=2$, $y=5$

When $x=-1$, $y=8$



From (i) and (ii), the points of intersection are:

$\left(\dfrac{-1\pm\sqrt{65}}{2}, \dfrac{15\mp\sqrt{65}}{2}\right)$, $(\boxed{\mathbf{2}}, \boxed{\mathbf{5}})$, $(\boxed{\mathbf{-1}}, \boxed{\mathbf{8}})$

Answers: x^2-9, x^2-9, $-x^2+9$, $-x^2+9$ (in order) (2, 5), (−1, 8) (in any order)

L 38b

(1) $y=|x^2-2x-3|$, $y=x+3$

[Sol]

$x^2-2x-3=(x-3)(x+1)$

(i) When $x \leqslant -1$ or $x \geqslant 3$,

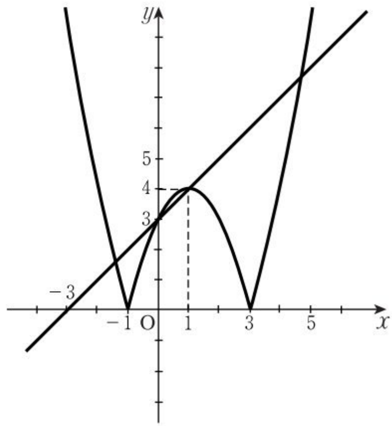
$$|x^2-2x-3|=x^2-2x-3$$

From $x^2-2x-3=x+3$,

$x^2-3x-6=0$, and so $x=\dfrac{3\pm\sqrt{33}}{2}$

When $x=\dfrac{3+\sqrt{33}}{2}$, $y=\dfrac{9+\sqrt{33}}{2}$

When $x=\dfrac{3-\sqrt{33}}{2}$, $y=\dfrac{9-\sqrt{33}}{2}$



(ii) When $-1<x<3$,

$$|x^2-2x-3|=-x^2+2x+3$$

From $-x^2+2x+3=x+3$,

$x^2-x=0$, and so $x=0,\ 1$

When $x=0$, $y=3$

When $x=1$, $y=4$

From (i) and (ii), the points of intersection are:

$$\left(\boldsymbol{\frac{3\pm\sqrt{33}}{2}},\ \boldsymbol{\frac{9\pm\sqrt{33}}{2}}\right),\ \mathbf{(0,\ 3),\ (1,\ 4)}$$

L 39a

KUMON

Modulus Functions

Ex.

Find the range of k for which the graphs of $y=|x^2-4|$ and $y=2x+k$ have exactly four common points.

[Sol]

When $x\leqslant -2$ or $x\geqslant 2$, $y=x^2-4$ ☞ $|x^2-4|=x^2-4$

When $-2<x<2$, $y=-x^2+4$ ☞ $|x^2-4|=-x^2+4$

From the graph, $y=2x+k$ must be between (a) and (b).

(i) Setting $y=-x^2+4$ equal to $y=2x+k$,

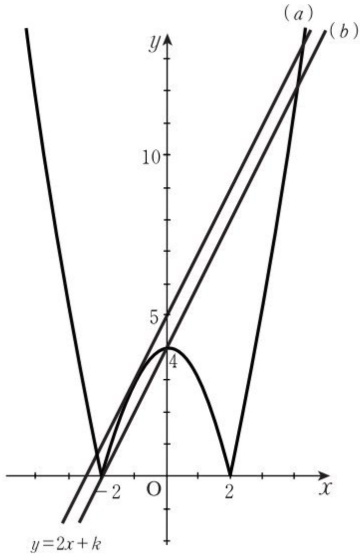
$$-x^2+4=\boxed{\mathbf{2x+k}}$$

$$x^2+2x+k-4=0$$

There is 1 common point when

$$\frac{D}{4}=1-(k-4)=\boxed{\mathbf{0}}$$

i.e. when $k=\boxed{\mathbf{5}}$



(ii) The line $y=2x+k$ passes through point $(-2,\ 0)$ when

$$0=2\times(-2)+k$$

i.e. when $k=\boxed{\mathbf{4}}$

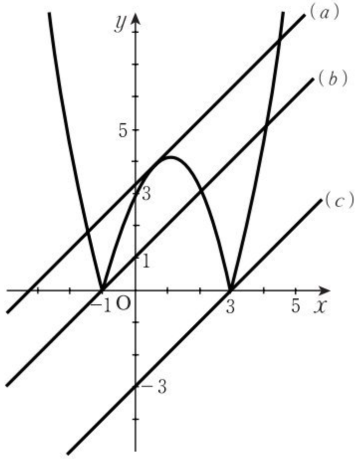
From (i) and (ii), $\boxed{\mathbf{4}}<k<\boxed{\mathbf{5}}$

Answers: in order $2x+k$, 0, 5, 4, 4, 5

L 39b

1. Find the range of k for which the graphs of $y=|(x+1)(x-3)|$ and $y=x+k$ have exactly two common points.

[Sol]



When $x \leqq -1$ or $x \geqq 3$, $y=(x+1)(x-3)$

When $-1<x<3$, $y=-(x+1)(x-3)$

From the graph, $y=x+k$ must be above (a) or between (b) and (c).

(i) Setting $y=-(x+1)(x-3)$ equal to $y=x+k$,

$$-(x+1)(x-3)=x+k$$

$$x^2-x+k-3=0$$

There is 1 common point when

$$D=1-4(k-3)=0,$$

i.e. when $k=\frac{13}{4}$

(ii) The line $y=x+k$ passes through $(-1,\ 0)$ when $0=-1+k$, i.e. when $k=1$

(iii) The line $y=x+k$ passes through $(3,\ 0)$ when $0=3+k$, i.e. when $k=-3$

From (i), (ii) and (iii), $\boldsymbol{-3<k<1,\ k>\frac{13}{4}}$

L 40a KUMON

Modulus Functions

1. For each function, write the letter (A) ~ (F) of the corresponding graph.

(1) $y=|x-2|$ ··· $\boxed{\textbf{(B)}}$

(2) $y=-|x+1|$ ··· $\boxed{\textbf{(F)}}$

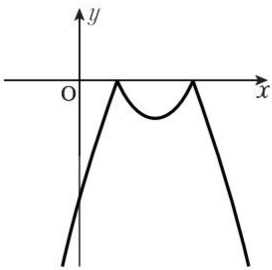
(3) $y=|(x-3)(x-1)|$ ··· $\boxed{\textbf{(C)}}$

(4) $y=|x^2+4x+3|$ ··· $\boxed{\textbf{(E)}}$

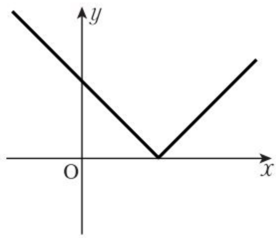
(5) $y=-|(x-3)(x-1)|$ ··· $\boxed{\textbf{(A)}}$

(6) $y=|x(x-1)(x-2)|$ ··· $\boxed{\textbf{(D)}}$

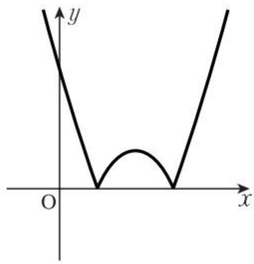
(A)



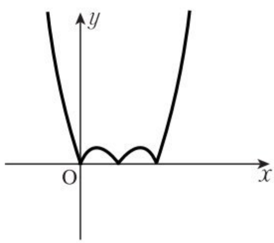
(B)



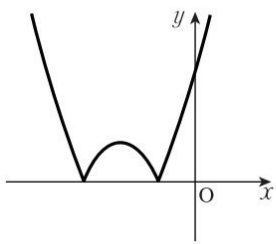
(C)



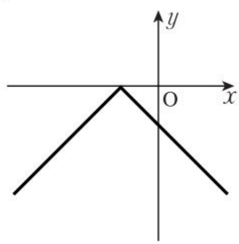
(D)



(E)



(F)



L 40b

2. Find the points of intersection of $y=|x^2-4|$ and $y=2x+1$.

[Sol]

$x^2-4=(x+2)(x-2)$

(i) When $x \leqslant -2$ or $x \geqslant 2$,

$|x^2-4|=x^2-4$

From $x^2-4=2x+1$,

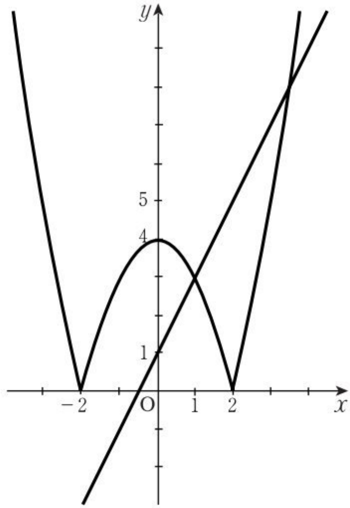
$x^2-2x-5=0$

$x=1\pm\sqrt{6}$

$x=1-\sqrt{6}$ is not in the domain.

When $x=1+\sqrt{6}$,

$y=2\times(1+\sqrt{6})+1=3+2\sqrt{6}$



(ii) When $-2<x<2$,

$|x^2-4|=-x^2+4$

From $-x^2+4=2x+1$,

$x^2+2x-3=0$

$x=-3, 1$

$x=-3$ is not in the domain.

When $x=1$,

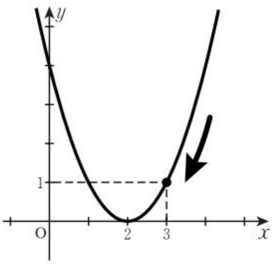
$y=2\times 1+1=3$

From (i) and (ii), the points of intersection are:

$(1+\sqrt{6}, 3+2\sqrt{6}), (1, 3)$

L 41 a KUMON

Limits and Derivatives



For function $y=(x-2)^2$, we write $\lim_{x\to 3}(x-2)^2$ to denote the ***limit*** value of the function, as the value of x approaches 3.
($\lim_{x\to 3} f(x)$ is read as "the limit, as x approaches 3, of $f(x)$".)

1. Find the following limits.

Ex.

$\lim_{x\to 3}(x-2)^2=(3-2)^2=1$ ☜ Substituting $x=3$.

(1) $\lim_{x\to 2} x^2=2^2=\mathbf{4}$

(2) $\lim_{x\to 0}(x^2+3x+3)=0^2+3\cdot 0+3=\mathbf{3}$

(3) $\lim_{x\to 0}\dfrac{x+3}{x+2}=\dfrac{0+3}{0+2}=\mathbf{\dfrac{3}{2}}$

(4) $\lim_{x\to -3}\dfrac{x^2-4}{x-2}=\dfrac{(-3)^2-4}{-3-2}=\dfrac{9-4}{-5}=\mathbf{-1}$

(5) $\lim_{x\to 0}(x^2+2a)=0^2+2a=\boldsymbol{2a}$

2. Find the following limits.

Ex.

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x+2)(x-2)}{x-2}=\lim_{x\to 2}(x+2)=4$$

Cancel out $x-2$. It is possible to do this as $x-2\neq 0$. [See note below.]

(1) $\displaystyle\lim_{x\to 3}\frac{x^2-9}{x-3}=\lim_{x\to 3}\frac{(x+3)(x-3)}{x-3}=\lim_{x\to 3}(x+3)=\mathbf{6}$

Cancel $x-3$ as $x\neq 3$.

(2) $\displaystyle\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{(x+1)(x-1)}{x-1}=\lim_{x\to 1}(x+1)=\mathbf{2}$

(3) $\displaystyle\lim_{x\to 2}\frac{(x-2)^2}{x-2}=\lim_{x\to 2}(x-2)=\mathbf{0}$

(4) $\displaystyle\lim_{x\to 1}\frac{x^2-x}{x-1}=\lim_{x\to 1}\frac{x(x-1)}{x-1}=\lim_{x\to 1}x=\mathbf{1}$

(5) $\displaystyle\lim_{x\to 0}\frac{2x-3x^3}{x-2x^2}=\lim_{x\to 0}\frac{x(2-3x^2)}{x(1-2x)}=\lim_{x\to 0}\frac{2-3x^2}{1-2x}=\mathbf{2}$

Note: The difference between $f(a)$ and $\lim_{x\to a}f(x)$

- $f(a)$ is the value of the function $f(x)$ when $x=a$.
- $\lim_{x\to a}f(x)$ is the limit value of $f(x)$ as x approaches a, but not equal to a.

L 42a KUMON

Limits and Derivatives

Find the following limits.

(1) $\lim\limits_{x \to -2} \dfrac{x^2-x-6}{x+2} = \lim\limits_{x \to -2} \dfrac{(x-3)(x+2)}{x+2} = \lim\limits_{x \to -2}(x-3) = \mathbf{-5}$

Cancel $x+2$ as $x \neq -2$.

(2) $\lim\limits_{x \to 2} \dfrac{x^2+2x-8}{x-2} = \lim\limits_{x \to 2} \dfrac{(x+4)(x-2)}{x-2} = \lim\limits_{x \to 2}(x+4) = \mathbf{6}$

(3) $\lim\limits_{x \to -1} \dfrac{x^3+1}{x+1} = \lim\limits_{x \to -1} \dfrac{(x+1)(x^2-x+1)}{x+1} = \lim\limits_{x \to -1}(x^2-x+1) = \mathbf{3}$

(4) $\lim\limits_{x \to 3} \dfrac{x-3}{x^2-9} = \lim\limits_{x \to 3} \dfrac{x-3}{(x+3)(x-3)} = \lim\limits_{x \to 3} \dfrac{1}{x+3} = \mathbf{\dfrac{1}{6}}$

(5) $\lim\limits_{x \to 1} \dfrac{x^3-x^2+x-1}{x^2+x-2} = \lim\limits_{x \to 1} \dfrac{(x-1)(x^2+1)}{(x+2)(x-1)} = \lim\limits_{x \to 1} \dfrac{x^2+1}{x+2} = \mathbf{\dfrac{2}{3}}$

L 42b

(6) $\lim_{h \to 0} \dfrac{2h+h^2}{h} = \lim_{h \to 0}(2+h) = \mathbf{2}$

Cancel h as $h \neq 0$.

(7) $\lim_{h \to 0} \dfrac{3h-3h^2+h^3}{h} = \lim_{h \to 0}(3-3h+h^2) = \mathbf{3}$

(8) $\lim_{h \to 0} \dfrac{2ah+h^2}{h} = \lim_{h \to 0}(2a+h) = \boldsymbol{2a}$

(9) $$\lim_{h \to 0} \frac{(x+h)^2-x^2}{h} = \lim_{h \to 0} \frac{x^2+2xh+h^2-x^2}{h}$$
$$= \lim_{h \to 0} \frac{2xh+h^2}{h} = \lim_{h \to 0}(2x+h) = \boldsymbol{2x}$$

(10) $$\lim_{h \to 0} \frac{(x+h)^3-x^3}{h} = \lim_{h \to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$$
$$= \lim_{h \to 0} \frac{3x^2h+3xh^2+h^3}{h}$$
$$= \lim_{h \to 0}(3x^2+3xh+h^2) = \boldsymbol{3x^2}$$

L 43a KUMON

Limits and Derivatives

1. Find the slope of line AB, where points A and B are on the graph of $f(x) = x^2$, and have the following coordinates:

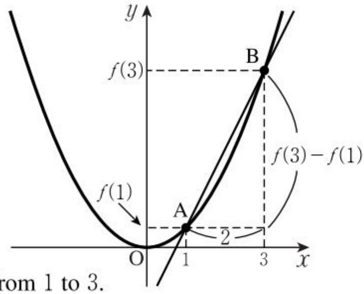
Ex.

The x-coordinate of point A is $x = 1$, and the x-coordinate of point B is 2 units greater.

[Sol] Thus, the x-coordinate of point B is $x = 3$.

Calculating the slope of line AB,

$$\frac{f(3) - f(1)}{3 - 1} = \frac{3^2 - 1^2}{2} = 4$$



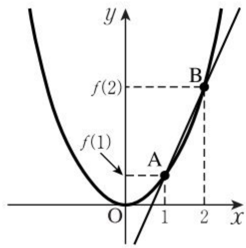
Note: The slope we have calculated is called the ***average rate of change*** of $f(x)$ as x varies from 1 to 3.

(1) The x-coordinate of point A is $x = 1$, and the x-coordinate of point B is 1 unit greater.

[Sol] Thus, the x-coordinate of B is $x = 2$.

Calculating the slope of AB,

$$\frac{f(2) - f(1)}{2 - 1} = \frac{2^2 - 1^2}{1} = \mathbf{3}$$

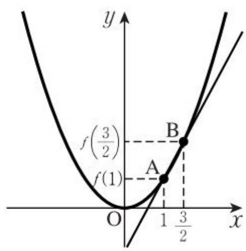


(2) The x-coordinate of point A is $x = 1$, and the x-coordinate of point B is $\frac{1}{2}$ units greater.

[Sol] Thus, the x-coordinate of B is $x = \frac{3}{2}$.

Calculating the slope of AB,

$$\frac{f\left(\frac{3}{2}\right) - f(1)}{\frac{3}{2} - 1} = \frac{\left(\frac{3}{2}\right)^2 - 1^2}{\frac{1}{2}} = \frac{\frac{5}{4}}{\frac{1}{2}} = \mathbf{\frac{5}{2}}$$



L 43b

2. On the graph of $f(x)=x^2$, the x-coordinate of point A is $x=1$, and the x-coordinate of point B is h units greater. Thus, the x-coordinate of point B is $x=1+h$. Therefore, the slope of line AB is:

$$\frac{f(1+h)-f(1)}{h}=\frac{(1+h)^2-1^2}{h}=\frac{2h+h^2}{h}=2+h$$ Cancel h as $h\neq 0$.

In each question, use the above formula to find the slope of line AB for each of the following values of h.

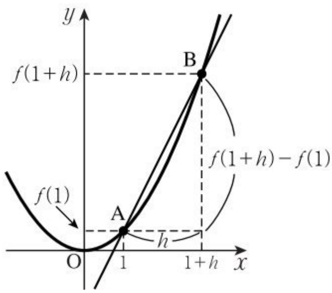
(1) When $h=1$, $2+\boxed{1}=\boxed{3}$

(2) When $h=0.5$, $\boxed{2.5}$

(3) When $h=0.1$, $\boxed{2.1}$

(4) When $h=0.01$, $\boxed{2.01}$

(5) When $h=0.001$, $\boxed{2.001}$

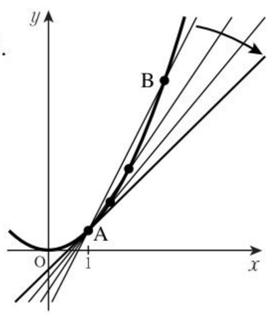


Note: From (1)～(5), we can conclude that:

- As h approaches zero, the slope of line AB approaches 2. This can be expressed as

$$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}(2+h)=2 \quad \cdots ①$$

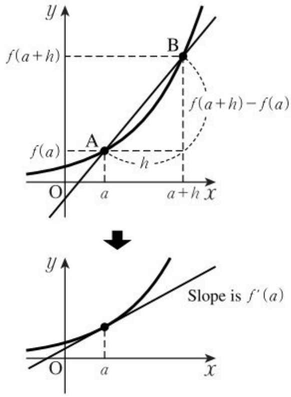
- The line that touches the curve at point A is called the ***tangent*** at point A, and point A is called the ***point of tangency***.
- ① indicates the slope of the tangent at point A.



L 44a KUMON

Limits and Derivatives

Given $y=f(x)$, the average rate of change as x varies from a to $a+h$ can be expressed as $\dfrac{f(a+h)-f(a)}{h}$. In this case,



$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

($f'(a)$ is read as "f prime of a".)

This is called the ***differential coefficient*** at $x=a$.

1. Given $f(x)=x^2$, find the following differential coefficient.

Ex.

$f'(2)$

[Sol] $f'(2)=\displaystyle\lim_{h\to 0}\frac{(2+h)^2-2^2}{h}$

$f(2+h)=(2+h)^2$
$f(2)=2^2$

$$=\lim_{h\to 0}\frac{4h+h^2}{h}$$

$$=\lim_{h\to 0}(4+h)=4$$

(1) $f'(-2)$

[Sol] $f'(-2)=\displaystyle\lim_{h\to 0}\frac{(-2+h)^2-(-2)^2}{h}$

$$=\lim_{h\to 0}\frac{-4h+h^2}{h}$$

$$=\lim_{h\to 0}(-4+h)$$

$$=\mathbf{-4}$$

The differential coefficient is also called the ***rate of change***.

L 44b

2. For each function $f(x)$, find the differential coefficient for each value of x shown.

Ex.

$f(x)=x^2-2x \quad (x=3)$

[Sol] $f'(3)=\lim_{h\to 0}\dfrac{[(3+h)^2-2(3+h)]-(3^2-2\cdot 3)}{h}$

$=\lim_{h\to 0}\dfrac{4h+h^2}{h}$

$=\lim_{h\to 0}(4+h)=4$

(1) $f(x)=(x-2)^2 \quad (x=-1)$

[Sol] $f'(-1)=\lim_{h\to 0}\dfrac{(-1+h-2)^2-(-1-2)^2}{h}$

$=\lim_{h\to 0}\dfrac{-6h+h^2}{h}$

$=\lim_{h\to 0}(-6+h)$

$=\mathbf{-6}$

(2) $f(x)=x^3-2x+4 \quad (x=2)$

[Sol] $f'(2)=\lim_{h\to 0}\dfrac{[(2+h)^3-2(2+h)+4]-(2^3-2\cdot 2+4)}{h}$

$=\lim_{h\to 0}\dfrac{10h+6h^2+h^3}{h}$

$=\lim_{h\to 0}(10+6h+h^2)$

$=\mathbf{10}$

L 45a KUMON

Limits and Derivatives

1. For the function $f(x)=x^2$, find the following differential coefficients.

(1) $f'(3)$

[Sol] $f'(3)=\lim\limits_{h\to 0}\dfrac{(3+h)^2-3^2}{h}$

$=\lim\limits_{h\to 0}\dfrac{6h+h^2}{h}$

$=\lim\limits_{h\to 0}(6+h)$

$=\mathbf{6}$

(2) $f'(0)$

[Sol] $f'(0)=\lim\limits_{h\to 0}\dfrac{(0+h)^2-0}{h}$

$=\lim\limits_{h\to 0}\dfrac{h^2}{h}$

$=\lim\limits_{h\to 0}h$

$=\mathbf{0}$

(3) $f'(-1)$

[Sol] $f'(-1)=\lim\limits_{h\to 0}\dfrac{(-1+h)^2-(-1)^2}{h}$

$=\lim\limits_{h\to 0}\dfrac{-2h+h^2}{h}$

$=\lim\limits_{h\to 0}(-2+h)$

$=\mathbf{-2}$

L 45b

2. The differential coefficient of $f(x)=x^2$ at $x=a$ can be expressed using the following formula:

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{(a+h)^2-a^2}{h}=\lim_{h\to 0}\frac{2ah+h^2}{h}=2a$$

Find the following differential coefficients using the above formula.

(1) When $a=3$, $f'(3)=2\cdot 3=\mathbf{6}$

(2) When $a=0$, $f'(0)=2\cdot 0=\mathbf{0}$

(3) When $a=-1$, $f'(-1)=2\cdot(-1)=\mathbf{-2}$

(4) When $a=-3$, $f'(-3)=2\cdot(-3)=\mathbf{-6}$

3. Fill in the blank boxes.

Question (1) from side a, and (1) from this page, give the same result.

(i) Question (2) from side a, and $\boxed{\textbf{(2)}}$ from this page, give the same result.

(ii) Question (3) from side a, and $\boxed{\textbf{(3)}}$ from this page, give the same result.

In the above questions $f'(a)=2a$, where $f'(a)$ is a function with a as the variable.
If we replace a with x, then $f'(x)=2x$. This is called the ***derivative*** of $f(x)$.

L 46a KUMON

Limits and Derivatives

The derivative of $f(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

1. Obtain the derivative of each function using the above formula.

Ex.

$f(x) = x^2$

[Sol] $$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$
$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$
$$= \lim_{h \to 0} (2x + h)$$
$$= 2x$$

(1) $f(x) = 2x^2$

[Sol] $$\boldsymbol{f'(x)} = \lim_{h \to 0} \frac{2(x+h)^2 - 2x^2}{h}$$
$$= \lim_{h \to 0} \frac{4xh + 2h^2}{h}$$
$$= \lim_{h \to 0} (4x + 2h)$$
$$\boldsymbol{= 4x}$$

Note: Derivatives can also be known as differentials or derived functions.

L 46b

(2) $f(x)=x^2+1$

[Sol] $\boldsymbol{f'(x)} = \lim\limits_{h\to 0} \dfrac{[(x+h)^2+1]-(x^2+1)}{h}$

$$= \lim_{h\to 0} \frac{2xh+h^2}{h}$$

$$= \lim_{h\to 0} (2x+h)$$

$$\boldsymbol{= 2x}$$

(3) $f(x)=x^2+2x$

[Sol] $\boldsymbol{f'(x)} = \lim\limits_{h\to 0} \dfrac{[(x+h)^2+2(x+h)]-(x^2+2x)}{h}$

$$= \lim_{h\to 0} \frac{2xh+h^2+2h}{h}$$

$$= \lim_{h\to 0} (2x+h+2)$$

$$\boldsymbol{= 2x+2}$$

Notation:

- $h=\Delta x$ is called the incremental change in x.
- $f(x+h)-f(x)=\Delta y$ is called the incremental change in y.

Using these symbols, the derivative can be expressed as:

$$f'(x) = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

(Δx is read as ***delta x***, and Δy is read as ***delta y***.)

($f'(x)$ is sometimes written as $\dfrac{dy}{dx}$, representing the limit value of $\dfrac{\Delta y}{\Delta x}$.)

L 47a KUMON

Limits and Derivatives

The process of finding a derivative is called ***differentiation***.
Differentiate the following functions.

(1) $f(x)=2x$

[Sol] $$\boldsymbol{f'(x)} = \lim_{h\to 0}\frac{2(x+h)-2x}{h}$$
$$= \lim_{h\to 0}\frac{2h}{h}$$
$$\mathbf{=2}$$

(2) $f(x)=x$

[Sol] $$\boldsymbol{f'(x)} = \lim_{h\to 0}\frac{(x+h)-x}{h}$$
$$= \lim_{h\to 0}\frac{h}{h}$$
$$\mathbf{=1}$$

(3) $f(x)=3$

[Sol] Since $f(x)=3$ for all values of x,
$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{3-3}{h}$$
$$= \mathbf{0}$$

L 47b

(4) $f(x)=x^3$

[Sol] $$\boldsymbol{f'(x)} = \lim_{h\to 0}\frac{(x+h)^3-x^3}{h}$$
$$= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}$$
$$= \lim_{h\to 0}(3x^2+3xh+h^2)$$
$$\boldsymbol{= 3x^2}$$

(5) $f(x)=x^4$

[Sol] $$f'(x) = \lim_{h\to 0}\frac{(x+h)^4-x^4}{h}$$
$$= \lim_{h\to 0}\frac{x^4+4x^3h+6x^2h^2+4xh^3+h^4-x^4}{h}$$
$$= \lim_{h\to 0}\frac{4x^3h+6x^2h^2+4xh^3+h^4}{h}$$
$$= \lim_{h\to 0}(4x^3+6x^2h+4xh^2+h^3)$$
$$= \boldsymbol{4x^3}$$

The derivative of $f(x)=x^2$ can be written as $(x^2)'$.
From the results of the example on L46 and questions (2), (4) and (5) on L47:

$(x)'=1$, $(x^2)'=2x$, $(x^3)'=3x^2$, $(x^4)'=4x^3$

Generally:

> When n is a natural number, $(x^n)'=nx^{n-1}$.

For example: $(x^7)'=7x^6$, $(x^{10})'=10x^9$

The natural numbers are 1, 2, 3, 4, 5, ...
i.e. the positive integers, or *counting numbers*.

L 48a KUMON

Limits and Derivatives

Given the function $y = f(x)$, the derivative $f'(x)$ can be expressed as y'.
Given two functions $f(x)$ and $g(x)$ and the constant k, we have the following rules of differentiation:

Rules of Differentiation

1. If $y = k$, then $y' = 0$
2. If $y = kf(x)$, then $y' = kf'(x)$
3. If $y = f(x) + g(x)$, then $y' = f'(x) + g'(x)$
4. If $y = f(x) - g(x)$, then $y' = f'(x) - g'(x)$

1. Differentiate the following functions.

Ex.

$y = 2x^3$

Since $(2x^3)' = 2(x^3)' = 2 \cdot 3x^2$

[Sol] $y' = 6x^2$

(1) $y = 5x^3$

[Sol] $\boldsymbol{y' = 15x^2}$

(2) $y = -2x^3$

[Sol] $\boldsymbol{y' = -6x^2}$

(3) $y = 7x^4$

[Sol] $\boldsymbol{y' = 28x^3}$

(4) $y = x^5$

[Sol] $\boldsymbol{y' = 5x^4}$

(5) $y = -4x^2$

[Sol] $\boldsymbol{y' = -8x}$

(6) $y = -3x$

[Sol] $\boldsymbol{y' = -3}$

(7) $y = 8$

[Sol] $\boldsymbol{y' = 0}$

(8) $y = ax^2$

[Sol] $\boldsymbol{y' = 2ax}$

L 48b

2. Differentiate the following functions.

Ex.

$y=2x^3-5x+2$

[Sol] $y'=6x^2-5$

Since $(2x^3)'=6x^2$, $(-5x)'=-5$ and $(2)'=0$

(1) $y=3x^2-5x-4$

[Sol] $\boldsymbol{y'=6x-5}$

(2) $y=2x^2-5x$

[Sol] $\boldsymbol{y'=4x-5}$

(3) $y=2x^4-3x^3+2x^2$

[Sol] $\boldsymbol{y'=8x^3-9x^2+4x}$

(4) $y=5-3x+4x^2-2x^5$

[Sol] $\boldsymbol{y'=-3+8x-10x^4}$

($\boldsymbol{y'=-10x^4+8x-3}$)

(5) $y=\dfrac{3}{4}x^4+x^2$

[Sol] $\boldsymbol{y'=3x^3+2x}$

(6) $y=ax^2+bx+c$

[Sol] $\boldsymbol{y'=2ax+b}$

L 49a KUMON

Limits and Derivatives

1. Differentiate the following functions.

(1) $y=(2x+3)^2=4x^2+12x+9$

[Sol] $\boldsymbol{y'=8x+12}$

(2) $y=(2x^2+3)(2x^2-3)=4x^4-9$

[Sol] $\boldsymbol{y'=16x^3}$

(3) $y=(x+2)^3=x^3+6x^2+12x+8$

[Sol] $\boldsymbol{y'=3x^2+12x+12}$

(4) $y=x(2x^3-3x+5)=2x^4-3x^2+5x$

[Sol] $\boldsymbol{y'=8x^3-6x+5}$

(5) $y=(x^2+1)(x^3+1)=x^5+x^3+x^2+1$

[Sol] $\boldsymbol{y'=5x^4+3x^2+2x}$

L 49b

Given two functions $f(x)$ and $g(x)$, we have a further rule of differentiation:

Rules of Differentiation

5. If $y=f(x)g(x)$, then $y'=f'(x)g(x)+f(x)g'(x)$

2. Differentiate the following functions using the rules of differentiation.

Ex.

$y=(x^2+1)(x^3+1)$ ☜ Let x^2+1 be $f(x)$; Let x^3+1 be $g(x)$.

[Sol] $y'=(x^2+1)'(x^3+1)+(x^2+1)(x^3+1)'$

$=2x(x^3+1)+(x^2+1)\cdot 3x^2$

$=2x^4+2x+3x^4+3x^2$

$=5x^4+3x^2+2x$

Note: The function in this example is the same as that in question (5) on side a. The method is different but the result is the same.

(1) $y=x^2(x^3+1)$

[Sol] $\boldsymbol{y'}=2x(x^3+1)+x^2\cdot 3x^2$

$=2x^4+2x+3x^4$

$\boldsymbol{=5x^4+2x}$

(2) $y=(x^2-2x)(x+1)$

[Sol] $\boldsymbol{y'}=(2x-2)(x+1)+(x^2-2x)\cdot 1$

$=2x^2-2+x^2-2x$

$\boldsymbol{=3x^2-2x-2}$

(3) $y=(2x^2+1)(3x^2-1)$

[Sol] $\boldsymbol{y'}=4x(3x^2-1)+(2x^2+1)\cdot 6x$

$=12x^3-4x+12x^3+6x$

$\boldsymbol{=24x^3+2x}$

L 50a KUMON

Limits and Derivatives

1. Differentiate the following functions.

(1) $y = 2x(x-2)(x+1) = (2x^2-4x)(x+1)$

[Sol] $\boldsymbol{y'} = (2x^2-4x)'(x+1)+(2x^2-4x)(x+1)'$
$= (4x-4)(x+1)+(2x^2-4x)\cdot 1$
$= 4x^2-4+2x^2-4x$
$\boldsymbol{= 6x^2-4x-4}$

(2) $y = x^2(3x^2+1)(3x+2) = (3x^4+x^2)(3x+2)$

[Sol] $\boldsymbol{y'} = (12x^3+2x)(3x+2)+(3x^4+x^2)\cdot 3$
$= 36x^4+24x^3+6x^2+4x+9x^4+3x^2$
$\boldsymbol{= 45x^4+24x^3+9x^2+4x}$

(3)* $y = (x+2)^3 = (x+2)^2(x+2) = (x^2+4x+4)(x+2)$

[Sol] $\boldsymbol{y'} = (2x+4)(x+2)+(x^2+4x+4)\cdot 1$
$= 2x^2+8x+8+x^2+4x+4$
$\boldsymbol{= 3x^2+12x+12}$

L 50b

2. Differentiate the following functions. (You may use any method.)

(1) $y = ax^3 + bx^2 + cx + d$

[Sol] $\boldsymbol{y' = 3ax^2 + 2bx + c}$

(2) $y = x(x-a)^2 = x(x^2 - 2ax + a^2)$

[Sol] $y' = 1 \cdot (x^2 - 2ax + a^2) + x(2x - 2a)$
$= x^2 - 2ax + a^2 + 2x^2 - 2ax$
$\boldsymbol{= 3x^2 - 4ax + a^2}$

Alternative Solution
$y = x(x^2 - 2ax + a^2)$
$= x^3 - 2ax^2 + a^2x$
$\boldsymbol{y' = 3x^2 - 4ax + a^2}$

(3) $y = (x^2 - x)(x^2 - 4)$

[Sol] $y' = (2x - 1)(x^2 - 4) + (x^2 - x) \cdot 2x$
$= 2x^3 - 8x - x^2 + 4 + 2x^3 - 2x^2$
$\boldsymbol{= 4x^3 - 3x^2 - 8x + 4}$

Alternative Solution
$y = x^4 - x^3 - 4x^2 + 4x$
$\boldsymbol{y' = 4x^3 - 3x^2 - 8x + 4}$

(4) $y = (x+2)(x-5)(x-1) = (x^2 - 3x - 10)(x - 1)$

[Sol] $y' = (2x - 3)(x - 1) + (x^2 - 3x - 10) \cdot 1$
$= 2x^2 - 5x + 3 + x^2 - 3x - 10$
$\boldsymbol{= 3x^2 - 8x - 7}$

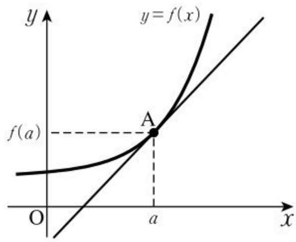
Alternative Solution
$y = (x^2 - 3x - 10)(x - 1)$
$= x^3 - x^2 - 3x^2 + 3x - 10x + 10$
$= x^3 - 4x^2 - 7x + 10$
$\boldsymbol{y' = 3x^2 - 8x - 7}$

L 51 a KUMON

Tangents

The gradient of the tangent to the curve $y=f(x)$ at the point $(a,\ f(a))$ is $f'(a)$.
The equation of the tangent at point $(a,\ f(a))$ is as follows:

$$\boxed{y-f(a)=f'(a)(x-a)}$$



Find the tangents to the following curves at the given points. Then, graph the tangents.

Ex.

$y=x^2-4x+4 \quad (3,\ 1)$

[Sol] Let $f(x)=x^2-4x+4$

$f'(x)=2x-4$

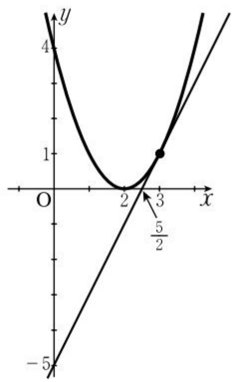
$f'(3)=2$

Find the gradient of the tangent.

Therefore, the tangent is:

$y-1=2(x-3)$

$y=2x-5$



(1) $y=-2x^2-4x+1 \quad (0,\ 1)$

[Sol] Let $f(x)=-2x^2-4x+1$

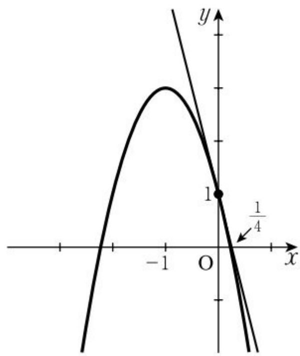
$f'(x)=-4x-4$

$f'(0)=-4$

The tangent is:

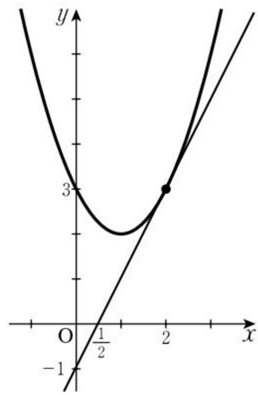
$y-1=-4(x-0)$

$\boldsymbol{y=-4x+1}$



L51b

(2) $y=x^2-2x+3$ $(2,\ 3)$



[Sol] Let $f(x)=x^2-2x+3$

$f'(x)=2x-2$

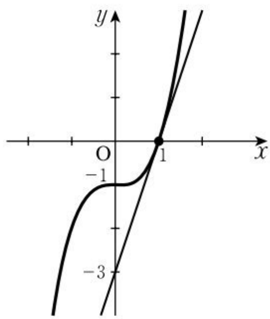
$f'(2)=2$

The tangent is:

$y-3=2(x-2)$

$\boldsymbol{y=2x-1}$

(3) $y=x^3-1$ $(1,\ 0)$



[Sol] Let $f(x)=x^3-1$

$f'(x)=3x^2$

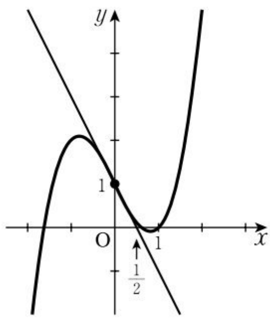
$f'(1)=3$

The tangent is:

$y-0=3(x-1)$

$\boldsymbol{y=3x-3}$

(4) $y=x^3-2x+1$ $(0,\ 1)$



[Sol] Let $f(x)=x^3-2x+1$

$f'(x)=3x^2-2$

$f'(0)=-2$

The tangent is:

$y-1=-2(x-0)$

$\boldsymbol{y=-2x+1}$

L 52a

KUMON

Tangents

Find the tangents to the following curves at the given points. Then, graph the tangents.

(1) $y = x^2 - 2x + 1 \quad (-1,\ 4)$

[Sol] Let $f(x) = x^2 - 2x + 1$

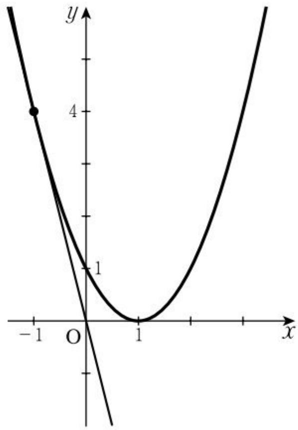
$f'(x) = 2x - 2$

$f'(-1) = -4$

The tangent is:

$y - 4 = -4(x+1)$

$\boldsymbol{y = -4x}$



(2) $y = x^3 - x^2 - x - 1 \quad (-1,\ -2)$

[Sol] Let $f(x) = x^3 - x^2 - x - 1$

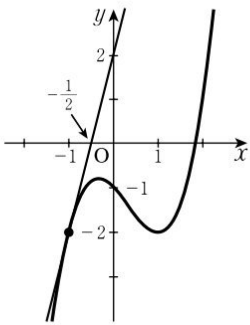
$f'(x) = 3x^2 - 2x - 1$

$f'(-1) = 4$

The tangent is:

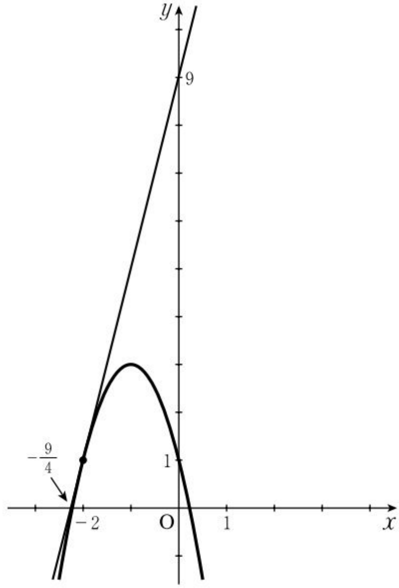
$y + 2 = 4(x+1)$

$\boldsymbol{y = 4x + 2}$



L 52b

(3) $y=-2x^2-4x+1 \quad (-2,\ 1)$



[Sol] Let $f(x)=-2x^2-4x+1$

$f'(x)=-4x-4$

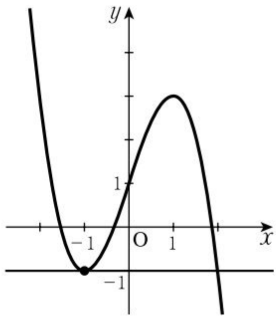
$f'(-2)=4$

The tangent is:

$y-1=4(x+2)$

$\boldsymbol{y=4x+9}$

(4) $y=-x^3+3x+1 \quad (-1,\ -1)$



[Sol] Let $f(x)=-x^3+3x+1$

$f'(x)=-3x^2+3$

$f'(-1)=0$

The tangent is:

$y+1=0\cdot(x+1)$

$\boldsymbol{y=-1}$

L 53a KUMON

Tangents

1. Find the tangent to each curve, at the point with the x-coordinate shown.

Ex.

$y = x^3 - 2x - 2 \quad (x = 1)$

[Sol] Let $f(x) = x^3 - 2x - 2$

$f'(x) = 3x^2 - 2$

$f'(1) = 1$

Find the gradient of the tangent.

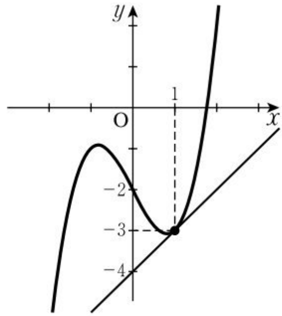
Also, $f(1) = -3$

The y-coordinate of the point of tangency

Therefore, the tangent is:

$y + 3 = 1 \cdot (x - 1)$

$y = x - 4$



(1) $y = x^3 - 2x - 2 \quad (x = -1)$

[Sol] Let $f(x) = x^3 - 2x - 2$

$f'(x) = 3x^2 - 2$

$f'(-1) = 1$

Also, $f(-1) = -1$

Therefore, the tangent is:

$y + 1 = 1 \cdot (x + 1)$

$\boldsymbol{y = x}$

(2) $y = x^2 - 2x + 4 \quad (x = 1)$

[Sol] Let $f(x) = x^2 - 2x + 4$

$f'(x) = 2x - 2$

$f'(1) = 0$

Also, $f(1) = 3$

Therefore, the tangent is:

$y - 3 = 0 \cdot (x - 1)$

$\boldsymbol{y = 3}$

L 53b

Ex.

Find the point of contact and the tangent to the curve $y = x^2 + 4x - 3$ where the tangent has gradient 2.

[Sol] Let $f(x) = x^2 + 4x - 3$

$f'(x) = 2x + 4$

From $2x + 4 = 2$,

Since the gradient is 2, let $f'(x) = 2$.

$x = -1$

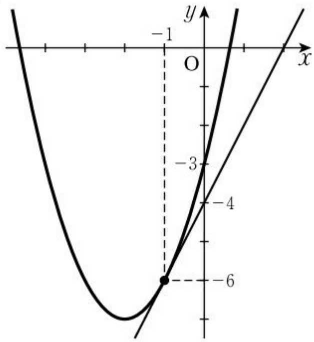
Also, $f(-1) = -6$,

so the point of contact is $(-1,\ -6)$.

Therefore, the tangent is:

$y + 6 = 2(x + 1)$

$y = 2x - 4$



2. Find the point of contact and the tangent to the curve $y = -2x^2 + 8x - 4$ where the tangent has gradient 4.

[Sol] Let $f(x) = -2x^2 + 8x - 4$

$f'(x) = -4x + 8$

From $-4x + 8 = 4$,

$x = 1$

Also, $f(1) = 2$,

so the point of contact is $\mathbf{(1,\ 2)}$.

Therefore, the tangent is:

$y - 2 = 4(x - 1)$

$\boldsymbol{y = 4x - 2}$

L 54a

Tangents

Ex.

Find the points of contact on the curve $y=x^3$ where the tangents have gradient 6.

[Sol] Let $f(x)=x^3$

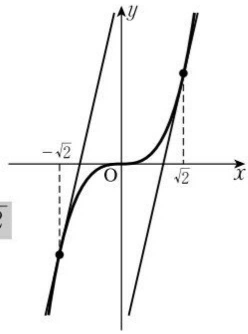
$f'(x)=3x^2$

From $3x^2=6$,

$x=\pm\sqrt{2}$

$f(\sqrt{2})=2\sqrt{2}$ The y-coordinate when $x=\sqrt{2}$

$f(-\sqrt{2})=-2\sqrt{2}$ The y-coordinate when $x=-\sqrt{2}$



Therefore, the points of contact are:

$(\sqrt{2},\ 2\sqrt{2}),\ (-\sqrt{2},\ -2\sqrt{2})$

1. Find the points of contact on the curve $y=\dfrac{1}{3}x^3-x^2+1$ where the tangents have gradient 3.

[Sol] Let $f(x)=\dfrac{1}{3}x^3-x^2+1$

$f'(x)=x^2-2x$

From $x^2-2x=3$,

$x=3,\ -1$

$f(3)=1$

$f(-1)=-\dfrac{1}{3}$

Therefore, the points of contact are:

$$\mathbf{(3,\ 1),\ \left(-1,\ -\frac{1}{3}\right)}$$

L 54b

2. Given the curve $y=x^3-2x+1$, complete the following questions.

(1) Find the points of contact where the tangents have gradient 1.

[Sol] Let $f(x)=x^3-2x+1$

$$f'(x)=3x^2-2$$

From $3x^2-2=1$,

$$x^2=1$$

$$x=\pm 1$$

$f(1)=0$, $f(-1)=2$

Therefore, the points of contact are:

$$\boldsymbol{(1,\ 0),\ (-1,\ 2)}$$

(2) Find the tangents passing through the points of contact found in (1).

[Sol] When the point of contact is $(1,\ 0)$, the tangent is:

$$y-0=1\cdot(x-1)$$

$$\boldsymbol{y=x-1}$$

When the point of contact is $(-1,\ 2)$, the tangent is:

$$y-2=1\cdot(x+1)$$

$$\boldsymbol{y=x+3}$$

L 55a

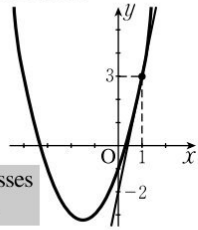
KUMON

Tangents

Ex.

Find the values of b and c for which the parabola $y = x^2 + bx + c$ has a tangent with gradient 5 at point $(1,\ 3)$.

[Reference]



[Sol] Let $f(x) = x^2 + bx + c$

$$f'(x) = 2x + b$$

Therefore,

$$\begin{cases} f(1) = 1 + b + c = 3 & \cdots① \\ f'(1) = 2 + b = 5 & \cdots② \end{cases}$$

① Since the parabola passes through point $(1,\ 3)$.

② Since the gradient is 5.

From ① and ②,

$$b = 3,\ c = -1$$

1. Find the values of a, b and c for which the parabola $y = ax^2 + bx + c$ passes through point $(1,\ 1)$, and has a tangent with gradient 7 at point $(3,\ 3)$.

[Sol] Let $f(x) = ax^2 + bx + c$

$$f'(x) = 2ax + b$$

Therefore,

$$\begin{cases} f(1) = a + b + c = 1 & \cdots① \\ f(3) = 9a + 3b + c = 3 & \cdots② \\ f'(3) = 6a + b = 7 & \cdots③ \end{cases}$$

From ①, ② and ③,

$$\boldsymbol{a = 3,\ b = -11,\ c = 9}$$

L 55b

2. Find the values of a, b and c for which the parabola $y = ax^2 + bx + c$ passes through point $(1,\ 3)$, and has the tangent $y = -4x + 9$ at point $(2,\ 1)$.

[Sol] Let $f(x) = ax^2 + bx + c$

$$f'(x) = 2ax + b$$

Therefore,

$$\begin{cases} f(1) = a + b + c = 3 & \cdots ① \\ f(2) = 4a + 2b + c = 1 & \cdots ② \\ f'(2) = 4a + b = -4 & \cdots ③ \end{cases}$$

From ①, ② and ③,

$$\boldsymbol{a = -2,\ b = 4,\ c = 1}$$

L 56a KUMON

Tangents

Ex.

The curve $y = x^3 + ax^2 + bx + c$ passes through point $(3,\ 9)$, and has a tangent with gradient 1 at point $(1,\ 7)$. Find a, b and c.

[Sol] Let $f(x) = x^3 + ax^2 + bx + c$

$$f'(x) = 3x^2 + 2ax + b$$

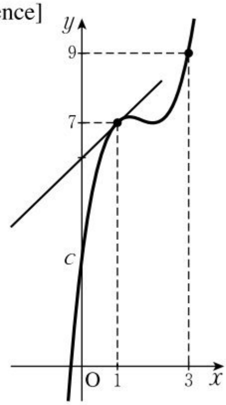
Therefore,

$$\begin{cases} f(3) = 27 + 9a + 3b + c = 9 & \cdots① \\ f(1) = 1 + a + b + c = 7 & \cdots② \\ f'(1) = 3 + 2a + b = 1 & \cdots③ \end{cases}$$

From ①, ② and ③,

$$a = -5,\ b = 8,\ c = 3$$

[Reference]



1. The curve $y = x^3 + ax^2 + bx + c$ passes through point $(0,\ -1)$, and has a tangent with gradient 2 at point $(1,\ 1)$. Find a, b and c.

[Sol] Let $f(x) = x^3 + ax^2 + bx + c$

$$f'(x) = 3x^2 + 2ax + b$$

Therefore,

$$\begin{cases} f(0) = c = -1 & \cdots① \\ f(1) = 1 + a + b + c = 1 & \cdots② \\ f'(1) = 3 + 2a + b = 2 & \cdots③ \end{cases}$$

From ①, ② and ③,

$$\boldsymbol{a = -2,\ b = 3,\ c = -1}$$

L56b

2. Find a, b, c and d, when the curve $y=ax^3+bx^2+cx+d$ touches the line $y=x+1$ at point $(0,\ 1)$, and the line $y=-2x+10$ at point $(3,\ 4)$.

[Sol] Let $f(x)=ax^3+bx^2+cx+d$

$$f'(x)=3ax^2+2bx+c$$

Therefore,

$$\begin{cases} f(0)=d=1 & \cdots① \\ f'(0)=c=1 & \cdots② \\ f(3)=27a+9b+3c+d=4 & \cdots③ \\ f'(3)=27a+6b+c=-2 & \cdots④ \end{cases}$$

From ①, ②, ③ and ④,

$$\boldsymbol{a=-\frac{1}{3},\ b=1,\ c=1,\ d=1}$$

L 57a KUMON

Tangents

Ex.

Given the curve $y=x^3$, find the tangent that touches the curve at point P$(1,\ 1)$. Then, find the point of intersection (other than point P) of the tangent with the curve.

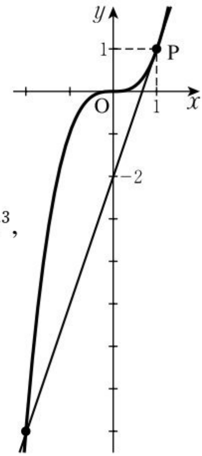
[Sol] Let $f(x)=x^3$

$$f'(x)=3x^2, \quad f'(1)=3$$

Therefore, finding the tangent at point P,

from $y-1=3(x-1)$,

$$y=3x-2 \quad \cdots①$$



Finding the point of intersection of ① and $y=x^3$,

from $x^3=3x-2$,

$$x^3-3x+2=0$$

$$(x-1)(x^2+x-2)=0$$

$$(x-1)^2(x+2)=0$$

$$x=1,\ \boxed{-2}$$

At $x=\boxed{-2}$, $f(\ -2\ )=\boxed{-8}$ ☜ The y-coordinate of the point of intersection

Therefore, the point of intersection (other than point P) is:

$(\ -2\ ,\ -8\ )$

Answers: in order −2, −2, −2, −8, −2, −8

L 57b

1. Given the curve $y=x^3-3x^2+x+1$, find the tangent that touches the curve at point $P(2,\ -1)$. Then, find the point of intersection (other than point P) of the tangent with the curve.

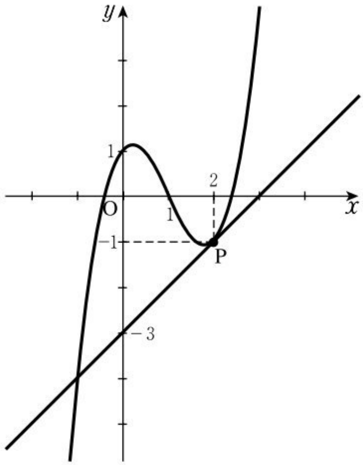
[Sol] Let $f(x)=x^3-3x^2+x+1$

$$f'(x)=3x^2-6x+1, \quad f'(2)=1$$

Therefore, finding the tangent at point P,

from $y+1=x-2$,

$$\boldsymbol{y=x-3} \quad \cdots ①$$



Finding the point of intersection of ① and $y=x^3-3x^2+x+1$,

from $x^3-3x^2+x+1=x-3$,

$$x^3-3x^2+4=0$$

$$(x-2)^2(x+1)=0$$

$$x=2,\ -1$$

At $x=-1$, $\quad f(-1)=-4$

Therefore, the point of intersection (other than point P) is:

$$\boldsymbol{(-1,\ -4)}$$

L 58a

Tangents

Ex.

Find the equations of the lines that pass through point $(1, -1)$ and are tangent to the curve $y = x^2 - x + 3$.

[Sol] Let a be the x-coordinate of a point of contact.

The point of contact is then $(a, a^2 - a + 3)$.

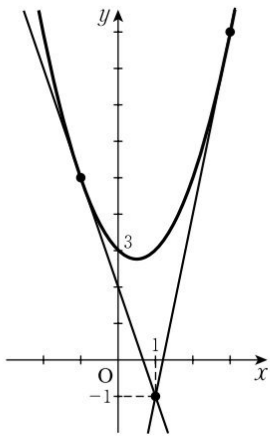
Let $f(x) = x^2 - x + 3$,

then from $f'(x) = 2x - 1$, $f'(a) = 2a - 1$

Therefore, the tangent is:

$$y - (a^2 - a + 3) = (2a - 1)(x - a)$$

$$y = (2a - 1)x - a^2 + 3 \quad \cdots ①$$



Since ① passes through $(1, -1)$,

$$-1 = 2a - 1 - a^2 + 3$$

$$a^2 - 2a - 3 = 0$$

$$a = 3, -1$$

Substituting $a = 3$ into ①,

$$y = \boxed{\mathbf{5x - 6}}$$

Substituting $a = -1$ into ①,

$$y = \boxed{\mathbf{-3x + 2}}$$

Answers: in order $5x - 6$, $-3x + 2$

L 58b

1. Find the equations of the lines that pass through point $(0,\ 0)$ and are tangent to the curve $y=x^2+3x+4$.

[Sol] Let a be the x-coordinate of a point of contact.

The point of contact is then $(a,\ a^2+3a+4)$.

Let $f(x)=x^2+3x+4$,

then from $f'(x)=2x+3$, $f'(a)=2a+3$

Therefore, the tangent is:

$$y-(a^2+3a+4)=(2a+3)(x-a)$$

$$y=(2a+3)x-a^2+4 \quad \cdots ①$$

Since ① passes through $(0,\ 0)$,

$$0=-a^2+4$$

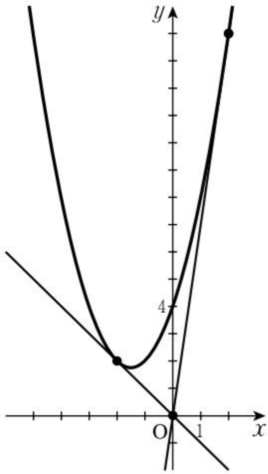
$$a=\pm 2$$

Substituting $a=2$ into ①,

$$\boldsymbol{y=7x}$$

Substituting $a=-2$ into ①,

$$\boldsymbol{y=-x}$$



L 59a KUMON

Tangents

1. Find the equations of the lines that pass through point $(-1,\ 3)$ and are tangent to the curve $y=x^3-2x+1$.

[Sol] Let a be the x-coordinate of a point of contact.

The point of contact is then $(a,\ a^3-2a+1)$.

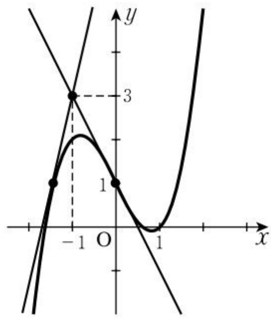
Let $f(x)=x^3-2x+1$,

then from $f'(x)=3x^2-2$, $f'(a)=3a^2-2$

Therefore, the tangent is:

$$y-(a^3-2a+1)=(3a^2-2)(x-a)$$

$$y=(3a^2-2)x-2a^3+1 \quad \cdots①$$



Since ① passes through $(-1,\ 3)$,

$$3=-3a^2+2-2a^3+1$$

$$2a^3+3a^2=0$$

$$a^2(2a+3)=0$$

$$a=0,\ -\frac{3}{2}$$

Substituting $a=0$ into ①,

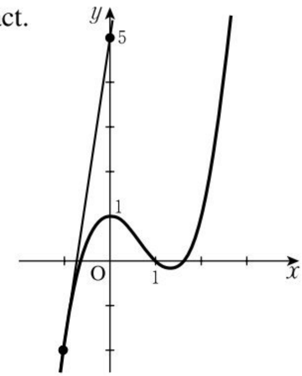
$$\boldsymbol{y=-2x+1}$$

Substituting $a=-\frac{3}{2}$ into ①,

$$\boldsymbol{y=\frac{19}{4}x+\frac{31}{4}}$$

L 59b

2. Find the equation of the line that passes through point $(0,\ 5)$ and is tangent to the curve $y = x^3 - 2x^2 + 1$.



[Sol] Let a be the x-coordinate of the point of contact.

The point of contact is then $(a,\ a^3 - 2a^2 + 1)$.

Let $f(x) = x^3 - 2x^2 + 1$,

then from $f'(x) = 3x^2 - 4x$, $f'(a) = 3a^2 - 4a$

Therefore, the tangent is:

$$y - (a^3 - 2a^2 + 1) = (3a^2 - 4a)(x - a)$$

$$y = (3a^2 - 4a)x - 2a^3 + 2a^2 + 1 \quad \cdots①$$

Since ① passes through $(0,\ 5)$,

$$5 = -2a^3 + 2a^2 + 1$$

$$2a^3 - 2a^2 + 4 = 0$$

$$a^3 - a^2 + 2 = 0$$

$$(a+1)(a^2 - 2a + 2) = 0$$

As $a^2 - 2a + 2 = (a-1)^2 + 1 > 0$,

there are no real solutions for $a^2 - 2a + 2 = 0$.

Therefore, $a = -1 \quad \cdots②$

Substituting ② into ①, the tangent is:

$$\boldsymbol{y = 7x + 5}$$

L 60a KUMON

Tangents

1. Find the equation of the line that has gradient 7 and is tangent to the curve $y = x^2 + 5x$.

[Sol] Let $f(x) = x^2 + 5x$

$$f'(x) = 2x + 5$$

From $2x + 5 = 7$,

$$x = 1$$

Also, $f(1) = 6$

Therefore, the point of contact is $(1,\ 6)$.

Thus, the tangent is:

$$y - 6 = 7(x - 1)$$

$$\boldsymbol{y = 7x - 1}$$

2. The curve $y = ax^2 + bx$ has a tangent with gradient 5 at point $(2,\ 2)$. Find a and b.

[Sol] Let $f(x) = ax^2 + bx$

$$f'(x) = 2ax + b$$

Therefore,

$$\begin{cases} f(2) = 4a + 2b = 2 & \cdots ① \\ f'(2) = 4a + b = 5 & \cdots ② \end{cases}$$

From ① and ②,

$$\boldsymbol{a = 2,\ b = -3}$$

L 60b

3. Find the equations of the lines that pass through the origin and are tangent to the curve $y=x^3-3x^2-1$.

[Sol] Let a be the x-coordinate of a point of contact.

The point of contact is then $(a,\ a^3-3a^2-1)$.

Let $f(x)=x^3-3x^2-1$,

then from $f'(x)=3x^2-6x$, $f'(a)=3a^2-6a$

Therefore, the tangent is:

$$y-(a^3-3a^2-1)=(3a^2-6a)(x-a)$$

$$y=(3a^2-6a)x-2a^3+3a^2-1 \quad \cdots①$$

Since ① passes through the origin,

$$0=-2a^3+3a^2-1$$

$$2a^3-3a^2+1=0$$

$$(a-1)^2(2a+1)=0$$

$$a=1,\ -\frac{1}{2}$$

Substituting $a=1$ into ①,

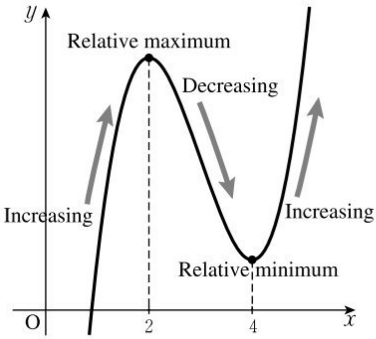
$$\boldsymbol{y=-3x}$$

Substituting $a=-\frac{1}{2}$ into ①,

$$\boldsymbol{y=\frac{15}{4}x}$$

L61a KUMON

Relative Maxima and Minima I



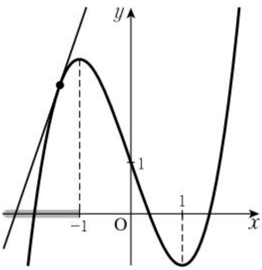
The cubic function
$f(x)=x^3-9x^2+24x-15$ is:

- increasing when $x<2$
- decreasing when $2<x<4$
- increasing when $x>4$

Also:

- At $x=2$, the function has a relative maximum.
- At $x=4$, the function has a relative minimum.

1. Complete the following questions on the function $f(x)=x^3-3x+1$ and its derivative $f'(x)=3x^2-3$. First circle the correct term from inside the brackets { }. Then, fill in the box with the appropriate symbol, $>$, $<$ or $=$.

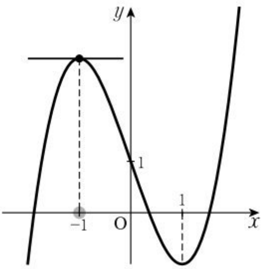


① When $x<-1$

(i) $f(x)$ is { increasing · decreasing }

(ii) Since the gradient of the tangent is positive,
$f'(x)\boxed{>}0$.

(For example, substituting $x=-2$ into $f'(x)$:
$f'(-2)=3\cdot(-2)^2-3=9>0$)

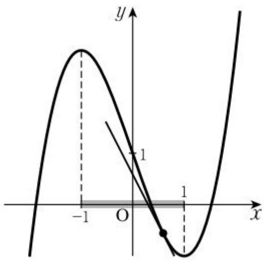


② When $x=-1$

(i) $f(x)$ has a { relative maximum · relative minimum }

(ii) Since the tangent is parallel to the x-axis, $f'(x)\boxed{=}0$.

Reminder: Horizontal lines, including the x-axis, have gradient 0, $(m=0)$.
Vertical lines, including the y-axis, have no gradient, (m is undefined).

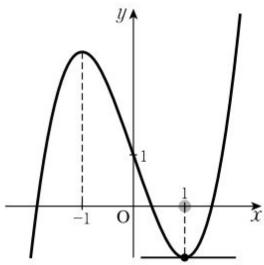


③ When $-1<x<1$

(i) $f(x)$ is { increasing · decreasing }

(ii) Since the gradient of the tangent is negative,

$f'(x)$ $\boxed{<}$ 0.

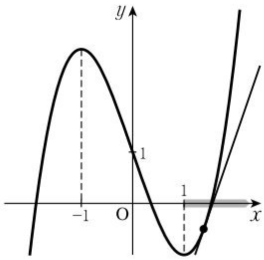


④ When $x=1$

(i) $f(x)$ has a $\left\{ \begin{array}{c} \text{relative} \\ \text{maximum} \end{array} \cdot \begin{array}{c} \text{relative} \\ \text{minimum} \end{array} \right\}$

(ii) Since the tangent is parallel to the

x-axis, $f'(x)$ $\boxed{=}$ 0.



⑤ When $x>1$

(i) $f(x)$ is { increasing · decreasing }

(ii) Since the gradient of the tangent is positive,

$f'(x)$ $\boxed{>}$ 0.

2. Complete the following using the graphs of questions 1. ①～⑤.

(i) From ① and ⑤, $f(x)$ is $\boxed{\textbf{increasing}}$ for values of x where $f'(x)>0$.

(ii) From ③, $f(x)$ is $\boxed{\textbf{decreasing}}$ for values of x where $f'(x)<0$.

(iii) From ② and ④, $f(x)$ has a $\boxed{\textbf{relative maximum}}$ or a

$\boxed{\textbf{relative minimum}}$ for values of x where $f'(x)=0$.

Note: When $f'(x)=0$, we do not know if $f(x)$ will be the relative maximum or the relative minimum.

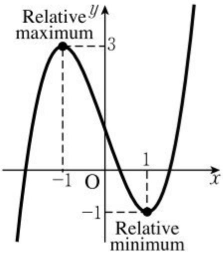
L 62a

Relative Maxima and Minima I

At the relative maximum, the value of $f(x)$ is called the “relative maximum value”, and at the relative minimum, the value of $f(x)$ is called the “relative minimum value”.

Ex.

Referring to the graph, find the values of x at which $f(x)=x^3-3x+1$ has a relative maximum and a relative minimum, then find the relative maximum value and the relative minimum value.



[Sol] $f'(x)=3x^2-3=3(x+1)(x-1)$

From $3(x+1)(x-1)=0$,

$x=-1,\ 1$

When $x=-1,\ f(-1)=3$

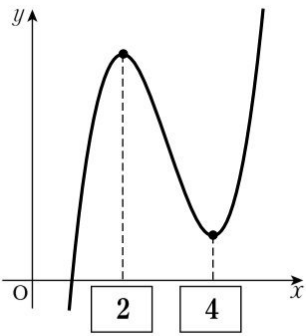
When $x=1,\ f(1)=-1$

To find the relative maximum and relative minimum, let $f'(x)=0$.

From the graph:

- At $x=-1$, the function has a relative maximum, and the relative maximum value is 3.
- At $x=1$, the function has a relative minimum, and the relative minimum value is -1.

1. Referring to the graph, find the values of x at which $f(x)=x^3-9x^2+24x-15$ has a relative maximum and a relative minimum, then find the relative maximum value and the relative minimum value.



[Sol] $f'(x)=3x^2-18x+24=\mathbf{3(x-4)(x-2)}$

From $3(x-4)(x-2)=0,\ x=2,\ 4$

When $x=2,\ f(2)=5$

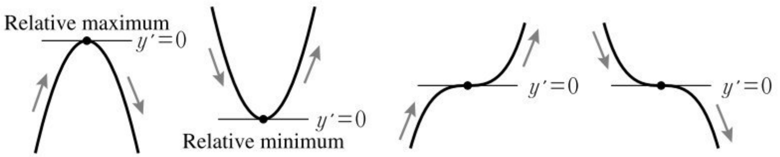
When $x=4,\ f(4)=1$

From the graph:

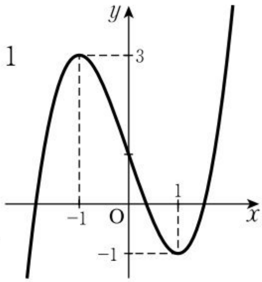
- At $x=\boxed{2}$, the function has a relative maximum, and the relative maximum value is $\boxed{5}$.
- At $x=\boxed{4}$, the function has a relative minimum, and the relative minimum value is $\boxed{1}$.

L 62b

We cannot identify whether a function has a relative maximum or a relative minimum, until we find out the variation around the point where $y' = 0$ (i.e. where $f'(x) = 0$).



2. Given the function $y = x^3 - 3x + 1$, find when it is increasing and decreasing, and find the relative extreme values (the relative maximum value and the relative minimum value).



[Sol] From $y' = 3x^2 - 3 = 3(x+1)(x-1) = 0$, $x = -1$, 1

(i) When $x < -1$, $y' > 0$, so y is increasing.

(ii) When $x = -1$, $y' = 0$ and $y = 3$.

(iii) When $-1 < x < 1$, $y' < 0$, so y is decreasing.

(iv) When $x = \boxed{\mathbf{1}}$, $y' = \boxed{\mathbf{0}}$ and $y = \boxed{\mathbf{-1}}$.

(v) When $x > 1$, $y' > 0$, so y is increasing.

At $x = -1$, y' changes from $y' > 0$ to $y' < 0$.

Thus, there is a relative maximum.

At $x = \boxed{\mathbf{1}}$, y' changes from $y' < 0$ to $y' > 0$.

Thus, there is a relative minimum.

The variation table (Summarises this information.)

x	$\cdots$	-1	$\cdots$	**1**	$\cdots$
y'	$+$	0	$-$	**0**	$+$
y	$\nearrow$	3	$\searrow$	**−1**	$\nearrow$
		(relative maximum)		(relative minimum)	

☜ We use $\cdots$ for the intervals of x in (i), (iii) and (v).

☜ When $y' > 0$ we use $+$ and when $y' < 0$ we use $-$.

☜ $\nearrow$ means increasing, $\searrow$ means decreasing.

Therefore, the relative maximum value is 3, at $x = -1$,

and the relative minimum value is $\boxed{\mathbf{-1}}$, at $x = \boxed{\mathbf{1}}$.

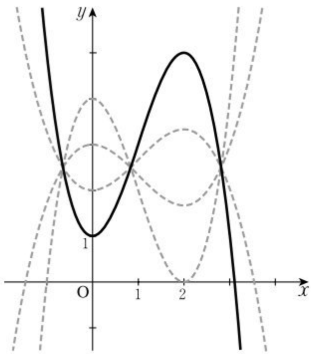
On the table, there is a relative maximum value when the arrow changes from $\nearrow$ to $\searrow$, and there is a relative minimum value when the arrow changes from $\searrow$ to $\nearrow$.

L 63a KUMON

Relative Maxima and Minima I

Ex.

Create a variation table for $y=-x^3+3x^2+1$, and find the relative maximum value and the relative minimum value. Then, trace the function on the graph provided.



[Sol] From $y'=-3x^2+6x=-3x(x-2)=0$,

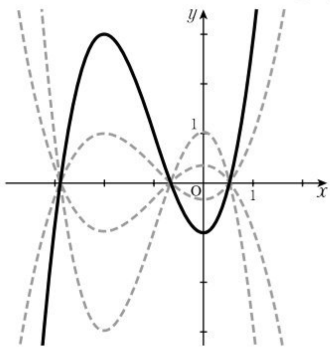
$x=0,\ 2$

Therefore, the variation table is:

x	$\cdots$	0	$\cdots$	2	$\cdots$
y'	$-$	0	$+$	0	$-$
y	↘	1	↗	5	↘
		(relative minimum)		(relative maximum)	

Therefore, the relative maximum value is 5, at $x=2$, and the relative minimum value is 1, at $x=0$.

1. Create a variation table for $y=x^3+3x^2-1$, and find the relative maximum value and the relative minimum value. Then, trace the function on the graph provided.



[Sol] From $y'=3x^2+6x=3x(x+2)=0$,

$x=0\ ,-2$

x	$\cdots$	-2	$\cdots$	0	$\cdots$
y'	$+$	0	$-$	0	$+$
y	↗	3	↘	-1	↗

Therefore:

The relative maximum value is $\boxed{3}$, at $x=\boxed{-2}$.

The relative minimum value is $\boxed{-1}$, at $x=\boxed{0}$.

In the table in the example, to find the sign of y' in any given interval you should choose a number that is easy to calculate with. For example, when $0<x<2$ we can choose $x=1$. Substituting this value into y', we find $y'=(-3)\cdot(-1)=3>0$.

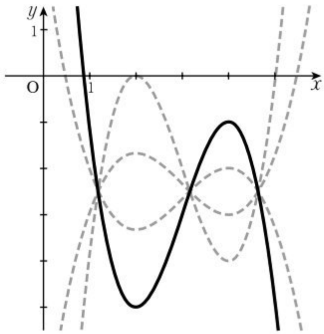
L 63b

2. Create a variation table for $y = -x^3 + 9x^2 - 24x + 15$, and find the relative maximum value and the relative minimum value. Then, trace the function on the graph provided.

[Sol] From $y' = -3x^2 + 18x - 24$

$= -3(x-4)(x-2) = 0$,

$x = 2,\ 4$



x	$\cdots$	**2**	$\cdots$	**4**	$\cdots$
y'	$-$	**0**	$+$	**0**	$-$
y	$\searrow$	$\mathbf{-5}$	$\nearrow$	$\mathbf{-1}$	$\searrow$

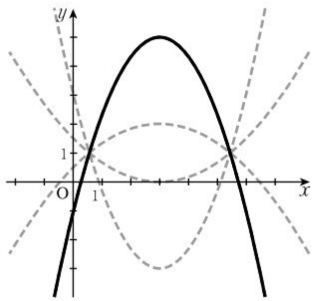
Therefore:

The relative maximum value is $\boxed{\mathbf{-1}}$, at $x = \boxed{\mathbf{4}}$.

The relative minimum value is $\boxed{\mathbf{-5}}$, at $x = \boxed{\mathbf{2}}$.

3. Create a variation table for $y = -\dfrac{2}{3}x^2 + 4x - 1$, and find the relative maximum value and the relative minimum value. Then, trace the function on the graph provided.

[Sol] From $y' = -\dfrac{4}{3}x + 4 = 0, \quad x = 3$



x	$\cdots$	**3**	$\cdots$
y'	$+$	**0**	$-$
y	$\nearrow$	**5**	$\searrow$

Therefore:

The relative maximum value is $\boxed{\mathbf{5}}$, at $x = \boxed{\mathbf{3}}$.

There is no relative minimum value.

Rearranging the equation in question 3. above by completing the square,

$y = -\dfrac{2}{3}x^2 + 4x - 1 = -\dfrac{2}{3}(x^2 - 6x) - 1 = -\dfrac{2}{3}(x-3)^2 + 5.$

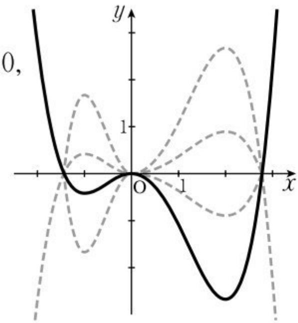
This is the method we previously used to find the vertex. However, given a quadratic function, we can also find the vertex by differentiating.

L 64a KUMON

Relative Maxima and Minima I

Ex.

Create a variation table for $y=\dfrac{1}{4}x^4-\dfrac{1}{3}x^3-x^2$, and find the relative maximum value(s) and the relative minimum value(s). Then, trace the function on the graph provided.



[Sol] From $y'=x^3-x^2-2x=x(x-2)(x+1)=0$,

$x=-1,\ 0,\ 2$

x	$\cdots$	-1	$\cdots$	0	$\cdots$	2	$\cdots$
y'	$-$	0	$+$	0	$-$	0	$+$
y	↘	$-\dfrac{5}{12}$	↗	0	↘	$-\dfrac{8}{3}$	↗

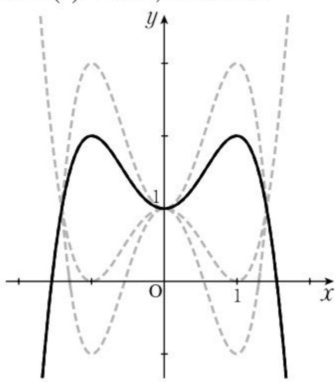
Therefore, we have:

A relative minimum value of $-\dfrac{5}{12}$, at $x=-1$.

A relative maximum value of 0, at $x=0$.

A relative minimum value of $-\dfrac{8}{3}$, at $x=2$.

1. Create a variation table for $y=-x^4+2x^2+1$, and find the relative maximum value(s) and the relative minimum value(s). Then, trace the function on the graph provided.



[Sol] From $y'=-4x^3+4x=-4x(x^2-1)$
$=-4x(x+1)(x-1)=0$,

$x=-1,\ 0,\ 1$

x	$\cdots$	-1	$\cdots$	0	$\cdots$	1	$\cdots$
y'	$+$	0	$-$	0	$+$	0	$-$
y	↗	2	↘	1	↗	2	↘

Therefore, we have:

A relative maximum value of $\boxed{2}$, at $x=\boxed{-1}$.

A relative minimum value of $\boxed{1}$, at $x=\boxed{0}$.

A relative maximum value of $\boxed{2}$, at $x=\boxed{1}$.

L 64b

2. Create a variation table for $y=x^4-4x^3+6x^2-4x+2$, and find the relative maximum value(s) and the relative minimum value(s). Then, trace the function on the graph provided.

[Sol] From $y'=4x^3-12x^2+12x-4$

$$=4(x^3-3x^2+3x-1)$$

$$=4(x-1)^3=0,$$

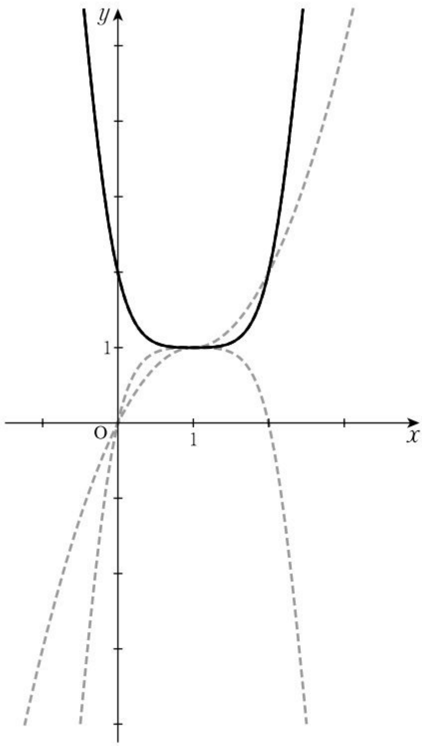
$x=1$

x	$\cdots$	**1**	$\cdots$
y'	$-$	**0**	$+$
y	$\searrow$	**1**	$\nearrow$

Therefore, we have:

No relative maximum value.

A relative minimum value of $\boxed{1}$, at $x=\boxed{1}$.



L 65a KUMON

Relative Maxima and Minima I

Ex.

Create a variation table for $y = x^3 + 3x^2 + 3x + 2$, and find any relative extreme values. Then, trace the function on the graph provided.

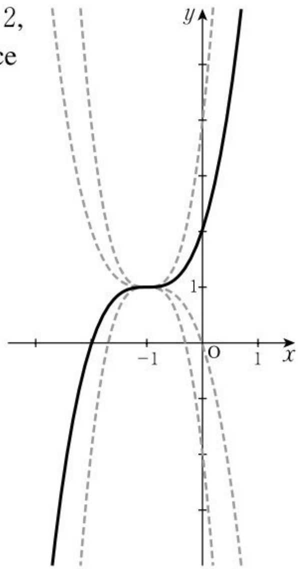
[Sol] From $y' = 3x^2 + 6x + 3 = 3(x+1)^2 = 0$,

$x = -1$

x	$\cdots$	-1	$\cdots$
y'	$+$	0	$+$
y	$\nearrow$	1	$\nearrow$

☞ See **Note**.

There are no relative extreme values.



Note: There are no relative extreme values when there are no arrows that change from $\nearrow$ to $\searrow$ or from $\searrow$ to $\nearrow$.

1. Create a variation table for $y = -x^3 - 3x^2 - 3x - 2$, and find any relative extreme values. Then, trace the function on the graph provided.

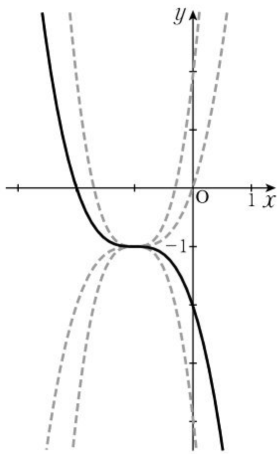
[Sol] From $y' = -3x^2 - 6x - 3$

$= -3(x+1)^2 = 0$,

$x = -1$

x	$\cdots$	-1	$\cdots$
y'	$-$	0	$-$
y	$\searrow$	-1	$\searrow$

There are no relative extreme values.



L 65b

2. Create a variation table for $y=\frac{1}{4}(x^4-6x^2-8x+12)$, and find any relative extreme values. Then, trace the function on the graph provided.

[Sol] From $y'=\frac{1}{4}(4x^3-12x-8)$

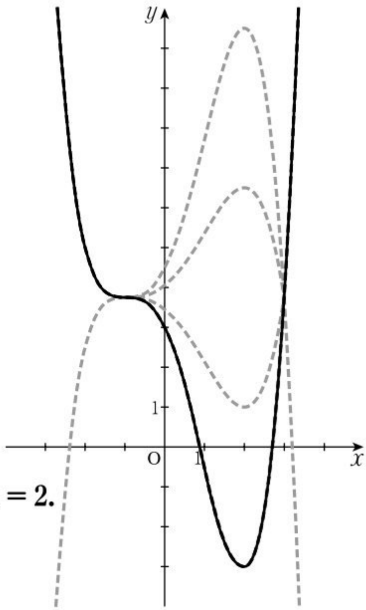
$$=x^3-3x-2$$
$$=(x+1)(x^2-x-2)$$
$$=(x+1)^2(x-2)=0,$$

$x=-1,\ 2$

x	$\cdots$	-1	$\cdots$	2	$\cdots$
y'	$-$	0	$-$	0	$+$
y	↘	$\frac{15}{4}$	↘	-3	↗

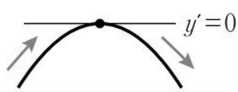
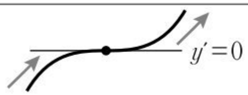
There is no relative maximum value.

The relative minimum value is -3, at $x=2$.



Note Summary

The relationship between the variation around the value of x at which $y'=0$, and the relative extreme values (relative maximum value or relative minimum value) can be summarised as follows:

When the arrows point from ↗ to ↘		$y'=0$ at this value of x and there is a relative maximum value.
When the arrows point from ↘ to ↗		$y'=0$ at this value of x and there is a relative minimum value.
When the arrows point from ↗ to ↗		$y'=0$ at this value of x but there is no relative extreme value.
When the arrows point from ↘ to ↘		$y'=0$ at this value of x but there is no relative extreme value.

L 66a KUMON

Relative Maxima and Minima I

1. Given $y=-x^3+x^2+x-1$, complete the questions below.

(1) Create a variation table to find the relative extreme values.

[Sol] From $y'=-3x^2+2x+1=\mathbf{-(3x+1)(x-1)}=0$,

$x=-\frac{1}{3},\ 1$

x	$\cdots$	$-\frac{1}{3}$	$\cdots$	1	$\cdots$
y'	$-$	0	$+$	0	$-$
y	$\searrow$	$-\frac{32}{27}$	$\nearrow$	0	$\searrow$

The relative maximum value is $\boxed{0}$, at $x=\boxed{1}$.

The relative minimum value is $\boxed{-\frac{32}{27}}$, at $x=\boxed{-\frac{1}{3}}$.

(2) Find the x-intercept(s) and y-intercept, then draw the graph.

① x-intercept(s)

From $-x^3+x^2+x-1=0$,

$-x^2(x-1)+(x-1)=0$

$-(x-1)^2(x+1)=0$

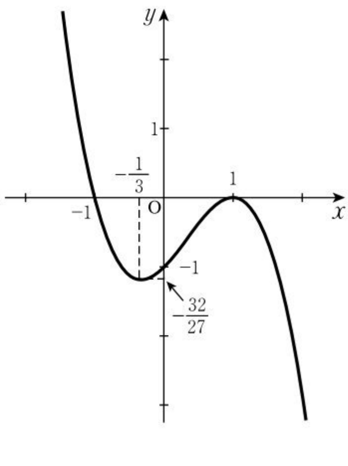
$x=1,\ -1$

Ans. $\mathbf{(1,\ 0),\ (-1,\ 0)}$

② y-intercept

Substituting $x=0$, $y=-1$

Ans. $\mathbf{(0,\ -1)}$



L 66b

2. Given $y=x^3+3x^2+3x+9$, complete the questions below.

(1) Create a variation table to find any relative extreme values.

[Sol] From $y'=3x^2+6x+3=3(x+1)^2=0$,

$x=-1$

x	$\cdots$	$\mathbf{-1}$	$\cdots$
y'	$+$	$\mathbf{0}$	$+$
y	$\nearrow$	$\mathbf{8}$	$\nearrow$

There are no relative extreme values.

(2) Find the x-intercept(s) and y-intercept, then draw the graph.

① x-intercept(s)

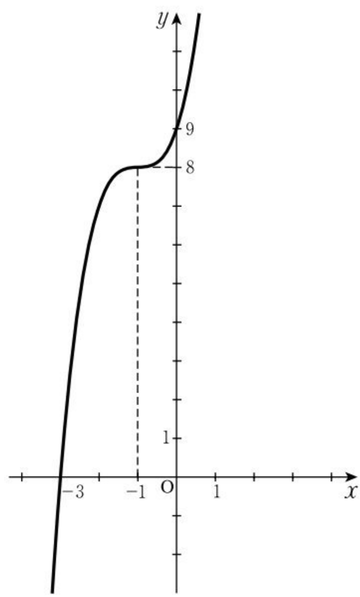
From $x^3+3x^2+3x+9=0$,

$x^2(x+3)+3(x+3)=0$

$(x+3)(x^2+3)=0$

$x=-3$

Ans. $\mathbf{(-3,\ 0)}$



② y-intercept

Substituting $x=0$, $y=9$

Ans. $\mathbf{(0,\ 9)}$

L 67a KUMON

Relative Maxima and Minima I

1. Given $y = x^4 - 2x^2 + 1$, complete the questions below.

(1) Create a variation table to find the relative extreme values.

[Sol] From $y' = 4x^3 - 4x = 4x(x+1)(x-1) = 0$,

$x = -1,\ 0,\ 1$

x	$\cdots$	-1	$\cdots$	0	$\cdots$	1	$\cdots$
y'	$-$	0	$+$	0	$-$	0	$+$
y	$\searrow$	0	$\nearrow$	1	$\searrow$	0	$\nearrow$

The relative minimum value is 0, at $x = -1$.
The relative maximum value is 1, at $x = 0$.
The relative minimum value is 0, at $x = 1$.

(2) Find the x-intercept(s) and y-intercept, then draw the graph.

① x-intercept(s)

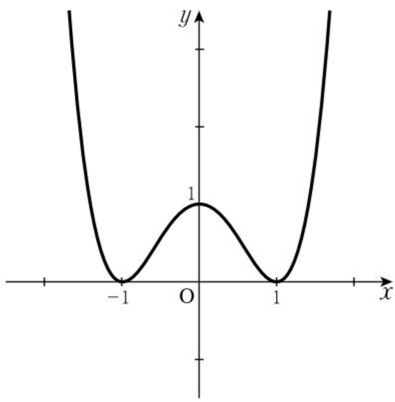
From $x^4 - 2x^2 + 1 = 0$,

$(x^2 - 1)^2 = 0$

$(x+1)^2(x-1)^2 = 0$

$x = -1,\ 1$

Ans. **$(-1,\ 0)$, $(1,\ 0)$**



② y-intercept

Substituting $x = 0$, $y = 1$

Ans. **$(0,\ 1)$**

L 67b

2. Given $y=\dfrac{1}{4}x^4-x^3$, complete the questions below.

(1) Create a variation table to find the relative extreme values.

[Sol] From $y'=x^3-3x^2=x^2(x-3)=0$,

$x=0,\ 3$

x	$\cdots$	**0**	$\cdots$	**3**	$\cdots$
y'	$-$	**0**	$-$	**0**	$+$
y	$\searrow$	**0**	$\searrow$	$\boldsymbol{-\dfrac{27}{4}}$	$\nearrow$

There is no relative maximum value.

The relative minimum value is $\boldsymbol{-\dfrac{27}{4}}$, at $\boldsymbol{x=3}$.

(2) Find the x-intercept(s) and y-intercept, then draw the graph.

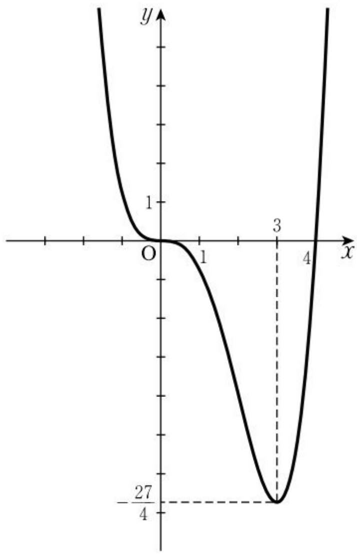
① x-intercept(s)

From $\dfrac{1}{4}x^4-x^3=0$,

$\dfrac{1}{4}x^3(x-4)=0$

$x=0,\ 4$

Ans. **(0, 0), (4, 0)**



② y-intercept

Substituting $x=0$, $y=0$

Ans. **(0, 0)**

L 68a KUMON

Relative Maxima and Minima I

For all values of x:

When $y' \geqslant 0$ (with $y' = 0$ at isolated points only), then y is increasing.

When $y' \leqslant 0$ (with $y' = 0$ at isolated points only), then y is decreasing.

In either case, there is no relative extreme value.

Ex.

Find the relative extreme values (if any) of $y = x^3 + 3x^2 + 6x + 9$.

[Sol] $y' = 3x^2 + 6x + 6$

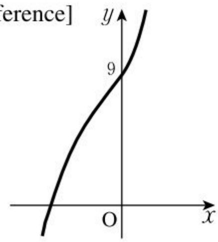
$= 3(x^2 + 2x + 2)$

$= 3(x+1)^2 + 3 > 0$ ☜ Here, for all values of x, $y' > 0$.

Therefore,

there are no relative extreme values.

[Reference]



1. Find the relative extreme values (if any) of $y = \frac{1}{3}x^3 - x^2 + 2x$.

[Sol] $y' = x^2 - 2x + 2$

$= (x-1)^2 + 1 > 0$

Therefore,

there are no relative extreme values.

2. Find the relative extreme values (if any) of $y = -x^3 + 3x^2 - 4x + 3$.

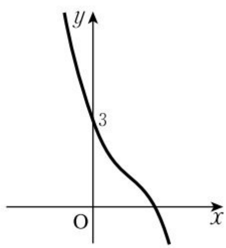
[Sol] $y' = -3x^2 + 6x - 4$

$= -3(x^2 - 2x) - 4$

$= -3(x-1)^2 - 1 < 0$

Therefore,

there are no relative extreme values.



Note: Compare the following functions.

(i) For $y = x^3$, $y' = 0$ at an isolated point only ($x = 0$). However, y increases as x increases, for all x.

(ii) For $y = 10$, $y' = 0$ at all points. Thus y is constant for all x.

From these examples, it can be seen that it is necessary to have the condition that $y' = 0$ at isolated points only.

L 68b

3. Find the values of a for which $y = x^3 + ax^2 + 3x + 1$ is always increasing.

[Sol] $y' = 3x^2 + 2ax + 3$

If $y' \geqslant 0$ (with $y' = 0$ at isolated points only), then y is always increasing. In this case, the discriminant of $3x^2 + 2ax + 3 = 0$ must be negative or zero.

Therefore,

$$\frac{D}{4} = a^2 - 9$$
$$= \boldsymbol{(a+3)(a-3) \leqslant 0}$$

Thus, $\boldsymbol{-3 \leqslant a \leqslant 3}$

4. Find the values of a for which $y = x^3 + ax^2 + 3ax + 2$ has a relative extreme value.

[Sol] $y' = 3x^2 + 2ax + 3a$

There is a relative extreme value when y' changes from positive to negative, or from negative to positive. In this case, $3x^2 + 2ax + 3a = 0$ must have 2 real solutions, which means that the discriminant must be positive.

Therefore,

$$\frac{D}{4} = a^2 - 9a$$
$$= \boldsymbol{a(a-9) > 0}$$

Thus, $\boldsymbol{a < 0,\ a > 9}$

- If y increases as x increases for all values of x, this is called "y increases monotonically" or "monotone increasing".
- If y decreases as x increases for all values of x, this is called "y decreases monotonically" or "monotone decreasing".

L 69a KUMON

Relative Maxima and Minima I

1. For each equation, write the letter (A)～(F) of the corresponding sketch.

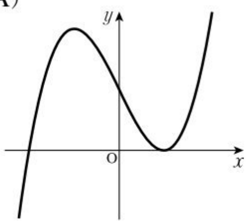
(1) $y=x^3-3x+2$ ··· $\boxed{\textbf{(A)}}$

(2) $y=-x^3+6x^2+2$ ··· $\boxed{\textbf{(D)}}$

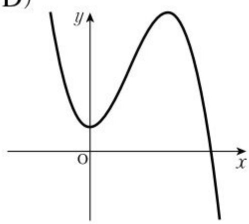
(3) $y=x^3+3x^2+3x+2$ ··· $\boxed{\textbf{(C)}}$

(4) $y=-x^3-3x$ ··· $\boxed{\textbf{(B)}}$

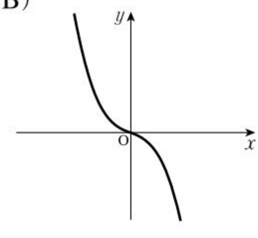
(A)



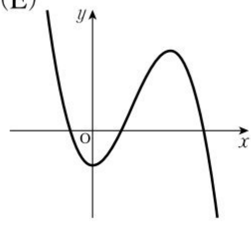
(D)



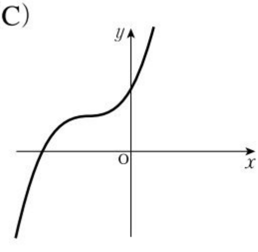
(B)



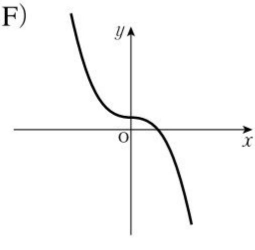
(E)



(C)



(F)



L 69b

2. For each equation, write the letter (A)～(F) of the corresponding sketch.

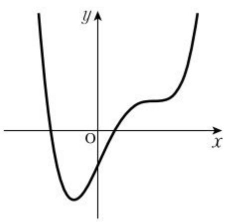
(1) $y=\dfrac{3}{4}x^4+x^3-3x^2-2$ ⋯ $\boxed{\textbf{(F)}}$

(2) $y=x^4-2$ ⋯ $\boxed{\textbf{(C)}}$

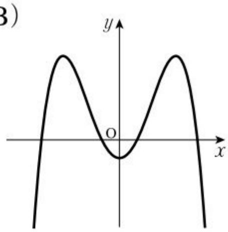
(3) $y=-x^4+8x^2-2$ ⋯ $\boxed{\textbf{(B)}}$

(4) $y=-x^4+4x^3-2$ ⋯ $\boxed{\textbf{(D)}}$

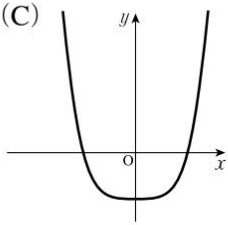
(A)



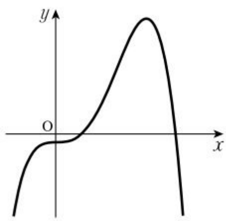
(B)



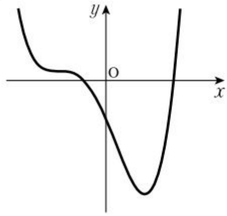
(C)



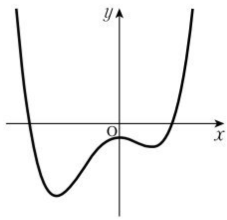
(D)



(E)



(F)



L 70a KUMON

Relative Maxima and Minima I

1. Given $y=x^3-6x^2+9x-4$, complete the questions below.

(1) Create a variation table to find the relative extreme values.

[Sol] From $y'=3x^2-12x+9$

$=3(x^2-4x+3)$

$=3(x-3)(x-1)=0$,

$x=1,\ 3$

x	$\cdots$	**1**	$\cdots$	**3**	$\cdots$
y'	$+$	**0**	$-$	**0**	$+$
y	$\nearrow$	**0**	$\searrow$	$\mathbf{-4}$	$\nearrow$

The relative maximum value is 0, at $x=1$.
The relative minimum value is -4, at $x=3$.

(2) Find the x-intercept(s) and y-intercept, then draw the graph.

① x-intercept(s)

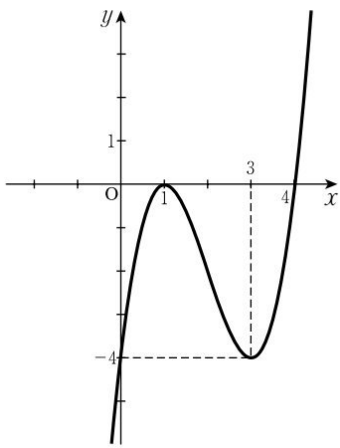
From $x^3-6x^2+9x-4=0$,

$(x-1)(x^2-5x+4)=0$

$(x-1)^2(x-4)=0$

$x=1,\ 4$

Ans. $\mathbf{(1,\ 0),\ (4,\ 0)}$



② y-intercept

Substituting $x=0,\ y=-4$

Ans. $\mathbf{(0,\ -4)}$

L 70b

2. Find the values of a for which $y=\frac{1}{3}x^3+ax^2+(a+6)x+3$ is always increasing.

[Sol] $y'=x^2+2ax+a+6$

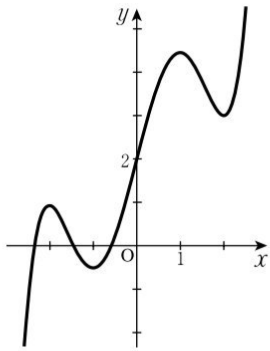
If $y'\geqslant 0$ (with $y'=0$ at isolated points only), then y is always increasing. In this case, the discriminant of $x^2+2ax+a+6=0$ must be negative or zero. Therefore,

$$\frac{D}{4}=a^2-(a+6)$$
$$=(a+2)(a-3)\leqslant 0$$

Thus, $\boldsymbol{-2\leqslant a\leqslant 3}$

Let's try this!

The 5th degree polynomial function $y=\frac{1}{5}x^5-\frac{5}{3}x^3+4x+2$ can be graphed as follows.



[Sol] $y'=x^4-5x^2+4$

$=(x^2-4)(x^2-1)$

$=(x+2)(x-2)(x+1)(x-1)$

At $y'=0$ the values of x are:

$x=\boxed{\mathbf{-2}}\,,\ \boxed{\mathbf{-1}}\,,\ \boxed{\mathbf{1}}\,,\ \boxed{\mathbf{2}}$

Therefore, setting up the variation table:

x	$\cdots$	$\boxed{\mathbf{-2}}$	$\cdots$	$\boxed{\mathbf{-1}}$	$\cdots$	$\boxed{\mathbf{1}}$	$\cdots$	$\boxed{\mathbf{2}}$	$\cdots$
y'	$+$	0	$-$	0	$+$	0	$-$	0	$+$
y	$\nearrow$	$\frac{14}{15}$	$\searrow$	$-\frac{8}{15}$	$\nearrow$	$\frac{68}{15}$	$\searrow$	$\frac{46}{15}$	$\nearrow$

Answers: in any order -2, -1, 1, 2, in order -2, -1, 1, 2

L71a KUMON

Relative Maxima and Minima II

Ex.

Given the function $f(x)=x^3+ax^2+bx-2$, when $x=1$ there is a relative maximum value of 2.

(1) Find the values of a and b.

[Sol] $f'(x)=3x^2+2ax+b$

From the given condition,

$f'(1)=3+2a+b=0 \quad \cdots①$

$f(1)=1+a+b-2=2 \quad \cdots②$

Since there is a relative maximum at $x=1$, let $f'(1)=0$.

From ① and ②,

$a=-6,\ b=9$

(2) Make a table showing that $f(x)$ has a relative maximum value of 2 at $x=1$, when a and b are the values found in (1). Then, find the relative minimum value.

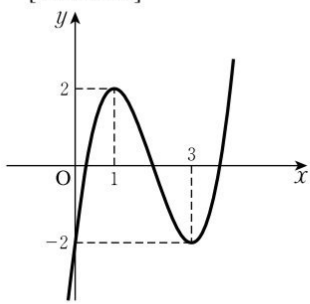
[Sol] $f(x)=x^3-6x^2+9x-2$

From $f'(x)=3x^2-12x+9=3(x-3)(x-1)=0$,

$x=1,\ 3$

x	$\cdots$	**1**	$\cdots$	**3**	$\cdots$
$f'(x)$	$+$	**0**	$-$	**0**	$+$
$f(x)$	↗	**2**	↘	**-2**	↗

[Reference]



Therefore:

At $x=1$,

there is a relative maximum value of 2.

And, at $x=$ $\boxed{3}$, there is a relative minimum value of $\boxed{-2}$.

Answers: in order

x	$\cdots$	1	$\cdots$	3	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	↗	2	↘	-2	↗

, 3, -2

L71b

1. Given the function $f(x)=x^3+ax^2+bx+12$, when $x=3$ there is a relative minimum value of -15.

(1) Find the values of a and b.

[Sol] $f'(x)=3x^2+2ax+b$

From the given condition,

$f'(3)=27+6a+b=0$ $\cdots$①

$f(3)=27+9a+3b+12=-15$ $\cdots$②

From ① and ②,

$\boldsymbol{a=-3,\ b=-9}$

(2) Make a table showing that $f(x)$ has a relative minimum value of -15 at $x=3$, when a and b are the values found in (1). Then, find the relative maximum value.

[Sol] $f(x)=x^3-3x^2-9x+12$

From $f'(x)=3x^2-6x-9=3(x-3)(x+1)=0$,

$x=-1,\ 3$

x	$\cdots$	$\mathbf{-1}$	$\cdots$	$\mathbf{3}$	$\cdots$
$f'(x)$	$+$	$\mathbf{0}$	$-$	$\mathbf{0}$	$+$
$f(x)$	$\nearrow$	$\mathbf{17}$	$\searrow$	$\mathbf{-15}$	$\nearrow$

Therefore:

At $x=3$, there is a relative minimum value of -15.

And, **at $\boldsymbol{x=-1}$, there is a relative maximum value of 17.**

L 72a KUMON

Relative Maxima and Minima II

1. Given that the function $f(x)=-x^3+ax^2+bx+2a$ has a relative maximum value of 3 at $x=1$, find a and b. Then, find the relative minimum value.

[Sol] $f'(x)=-3x^2+2ax+b$

From the given condition,

$f'(1)=-3+2a+b=0$ $\cdots$①

$f(1)=-1+a+b+2a=3$ $\cdots$②

From ① and ②,

$\boldsymbol{a=1,\ b=1}$

$f(x)=-x^3+x^2+x+2$

From $f'(x)=-3x^2+2x+1=-(3x+1)(x-1)=0$,

$x=-\dfrac{1}{3},\ 1$

x	$\cdots$	$-\frac{1}{3}$	$\cdots$	1	$\cdots$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\searrow$	$\frac{49}{27}$	$\nearrow$	3	$\searrow$

Therefore:

At $x=1$, there is a relative maximum value of 3.

And, **at $x=-\dfrac{1}{3}$, there is a relative minimum value of $\dfrac{49}{27}$.**

L 72b

2. Given that the function $f(x)=x^3+ax^2+bx+c$ passes through point $(1,\ 2)$, and that it has a relative minimum value of 0 at $x=2$, find a, b and c. Then, find the relative maximum value.

[Sol] $f'(x)=3x^2+2ax+b$

From the given conditions,

$f'(2)=12+4a+b=0 \quad \cdots①$

$f(2)=8+4a+2b+c=0 \quad \cdots②$

$f(1)=1+a+b+c=2 \quad \cdots③$

From ①, ② and ③,

$\boldsymbol{a=-3,\ b=0,\ c=4}$

$f(x)=x^3-3x^2+4$

From $f'(x)=3x^2-6x=3x(x-2)=0$,

$x=0,\ 2$

x	$\cdots$	0	$\cdots$	2	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	4	$\searrow$	0	$\nearrow$

Therefore:

At $x=2$, there is a relative minimum value of 0.

And, **at $x=0$, there is a relative maximum value of 4.**

L 73a KUMON

Relative Maxima and Minima II

1. Given the function $f(x)=ax^3+2ax^2+ax+1$, complete the following.

Ex.

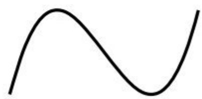
Find the relative maximum value of $f(x)$ when $a>0$.

[Sol] From $f'(x)=3ax^2+4ax+a=a(3x+1)(x+1)=0$,

$x=-1,\ -\dfrac{1}{3}$

x	$\cdots$	-1	$\cdots$	$-\frac{1}{3}$	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	relative maximum	$\searrow$	relative minimum	$\nearrow$

[Reference]
Since $a>0$ the graph is shaped:



$f(-1)=-a+2a-a+1=1$ ☜ Relative maximum at $x=-1$

Therefore,
the relative maximum value is 1, at $x=-1$.

(1) Find the relative maximum value of $f(x)$ when $a<0$.

[Sol]

x	$\cdots$	$\mathbf{-1}$	$\cdots$	$-\frac{\mathbf{1}}{\mathbf{3}}$	$\cdots$
$f'(x)$	$-$	$\mathbf{0}$	$+$	$\mathbf{0}$	$-$
$f(x)$	$\searrow$	**relative minimum**	$\nearrow$	**relative maximum**	$\searrow$

$f\left(-\dfrac{1}{3}\right)=-\dfrac{1}{27}a+\dfrac{2}{9}a-\dfrac{1}{3}a+1=-\dfrac{4}{27}a+1$

Therefore,
the relative maximum value is $-\dfrac{4}{27}a+1$, at $x=-\dfrac{1}{3}$.

L 73b

2. Given the function $f(x)=ax^3+3ax^2-9ax+1$, complete the following.

(1) Find the relative maximum value of $f(x)$ when $a>0$.

[Sol] From $f'(x)=3ax^2+6ax-9a=3a(x+3)(x-1)=0$,

$x=-3,\ 1$

x	$\cdots$	-3	$\cdots$	1	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	↗	relative maximum	↘	relative minimum	↗

$f(-3)=-27a+27a+27a+1=27a+1$

Therefore,

the relative maximum value is $\boldsymbol{27a+1}$, at $\boldsymbol{x=-3}$.

(2) Find the relative maximum value of $f(x)$ when $a<0$.

[Sol]

x	$\cdots$	-3	$\cdots$	1	$\cdots$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	↘	relative minimum	↗	relative maximum	↘

$f(1)=a+3a-9a+1=-5a+1$

Therefore,

the relative maximum value is $\boldsymbol{-5a+1}$, at $\boldsymbol{x=1}$.

Note: When the coefficient of x^3 is a letter, as in ax^3+2ax^2+ax+1, we find the relative extreme values by considering two cases (i) when $a>0$ and (ii) when $a<0$.

L 74a KUMON

Relative Maxima and Minima II

1. Find the relative minimum value of the function $f(x)=ax^3-3ax+4$, where $a\neq 0$. (Consider the cases where $a>0$ and $a<0$.)

[Sol] From $f'(x)=3ax^2-3a=3a(x+1)(x-1)=0$,

$x=-1,\ 1$

(i) When $a>0$,

x	$\cdots$	-1	$\cdots$	1	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	relative maximum	$\searrow$	relative minimum	$\nearrow$

$f(1)=a-3a+4=-2a+4$

(ii) When $a<0$,

x	$\cdots$	-1	$\cdots$	1	$\cdots$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\searrow$	relative minimum	$\nearrow$	relative maximum	$\searrow$

$f(-1)=-a+3a+4=2a+4$

From (i) and (ii):

$$\begin{cases} \textbf{When } \boldsymbol{a>0}\textbf{, the relative minimum value is } \boldsymbol{-2a+4}\textbf{, at } \boldsymbol{x=1}. \\ \textbf{When } \boldsymbol{a<0}\textbf{, the relative minimum value is } \boldsymbol{2a+4}\textbf{, at } \boldsymbol{x=-1}. \end{cases}$$

L 74b

2. Find the relative maximum value of the function $f(x)=ax^3-2ax^2-4ax+8$ (where $a\neq 0$).

[Sol] From $f'(x)=3ax^2-4ax-4a=a(3x+2)(x-2)=0$,

$$x=-\frac{2}{3},\ 2$$

(i) When $a>0$,

x	$\cdots$	$-\frac{2}{3}$	$\cdots$	2	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	↗	relative maximum	↘	relative minimum	↗

$$f\left(-\frac{2}{3}\right)=-\frac{8}{27}a-\frac{8}{9}a+\frac{8}{3}a+8=\frac{40}{27}a+8$$

(ii) When $a<0$,

x	$\cdots$	$-\frac{2}{3}$	$\cdots$	2	$\cdots$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	↘	relative minimum	↗	relative maximum	↘

$$f(2)=8a-8a-8a+8=-8a+8$$

From (i) and (ii):

$$\begin{cases}\textbf{When } a>0\textbf{, the relative maximum value is } \dfrac{40}{27}a+8\textbf{, at } x=-\dfrac{2}{3}. \\ \textbf{When } a<0\textbf{, the relative maximum value is } -8a+8\textbf{, at } x=2.\end{cases}$$

L 75a KUMON

Relative Maxima and Minima II

1. Given that the function $f(x)=ax^3+3ax^2-9ax$ (where $a\neq 0$) has a relative maximum value of 15, complete the following.

Ex.

Find the relative minimum value when $a>0$.

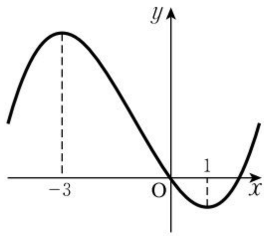
[Sol] From $f'(x)=3ax^2+6ax-9a=3a(x+3)(x-1)=0$,

$x=-3,\ 1$

$f(-3)=-27a+27a+27a=27a$

$f(1)=a+3a-9a=-5a$

x	$\cdots$	-3	$\cdots$	1	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	↗	relative maximum	↘	relative minimum	↗



From $f(-3)=27a=15$, $a=\dfrac{5}{9}$ ☜ Relative maximum at $x=-3$

Therefore, $f(1)=-5\cdot\dfrac{5}{9}=-\dfrac{25}{9}$

Thus, the relative minimum value is $-\dfrac{25}{9}$, at $x=1$.

(1) Find the relative minimum value when $a<0$.

[Sol]

x	$\cdots$	-3	$\cdots$	1	$\cdots$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	↘	relative minimum	↗	relative maximum	↘

From $f(1)=-5a=15$, $a=-3$

Therefore, $f(-3)=27\cdot(-3)=-81$

Thus, **the relative minimum value is -81, at $x=-3$.**

L 75b

2. Given that the function $f(x)=ax^3+9ax^2+24ax-4$ (where $a\neq 0$) has a relative maximum value of -20, find the relative minimum value.

[Sol] From $f'(x)=3ax^2+18ax+24a=3a(x+4)(x+2)=0$,

$x=-4, -2$

$f(-4)=-64a+144a-96a-4=-16a-4$

$f(-2)=-8a+36a-48a-4=-20a-4$

(i) When $a>0$,

x	$\cdots$	-4	$\cdots$	-2	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	relative maximum	$\searrow$	relative minimum	$\nearrow$

From $f(-4)=-16a-4=-20,\ a=1$

$f(-2)=-20\cdot 1-4=-24$

(ii) When $a<0$,

x	$\cdots$	-4	$\cdots$	-2	$\cdots$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\searrow$	relative minimum	$\nearrow$	relative maximum	$\searrow$

From $f(-2)=-20a-4=-20,\ a=\frac{4}{5}$

$a=\frac{4}{5}$ is inconsistent with $a<0$, thus, this is an extraneous solution.

From (i) and (ii):

The relative minimum value is -24, at $x=-2$.

L 76a KUMON

Relative Maxima and Minima II

Ex.

Find the relative minimum value of $y = x^2\left(x - \frac{3}{2}a\right)$.

[Sol] Let $f(x) = x^2\left(x - \frac{3}{2}a\right) = x^3 - \frac{3}{2}ax^2$

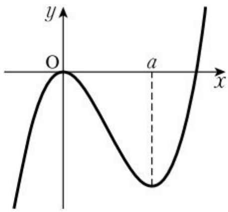
From $f'(x) = 3x^2 - 3ax = 3x(x-a) = 0$,

$x = 0,\ a$

$f(0) = 0,\ f(a) = -\frac{1}{2}a^3$

(i) When $a > 0$,

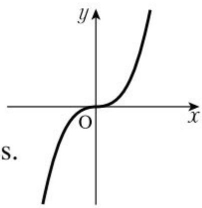
x	$\cdots$	0	$\cdots$	a	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	↗	relative maximum	↘	relative minimum	↗



(ii) When $a = 0$,

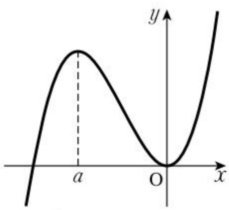
then from $f'(x) = 3x^2 \geqq \boxed{\mathbf{0}}$,

$f(x)$ does not have any relative extreme values.



(iii) When $a < 0$,

x	$\cdots$	$\boldsymbol{a}$	$\cdots$	**0**	$\cdots$
$f'(x)$	**+**	**0**	**−**	**0**	**+**
$f(x)$	↗	relative maximum	↘	relative minimum	↗



From (i), (ii) and (iii):

When $a > 0$, the relative minimum value is $\boxed{-\frac{1}{2}a^3}$, at $x = \boxed{a}$.

When $a = 0$, there is no relative minimum value.

When $a < 0$, the relative minimum value is $\boxed{\mathbf{0}}$, at $x = \boxed{\mathbf{0}}$.

Answers: in order 0,

x	$\cdots$	a	$\cdots$	0	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	↗	relative maximum	↘	relative minimum	↗

, $-\frac{1}{2}a^3$, a, 0, 0

L 76b

1. Find the relative maximum value of $y = x(x-a)^2$.

[Sol] Let $f(x) = x(x-a)^2 = x^3 - 2ax^2 + a^2x$

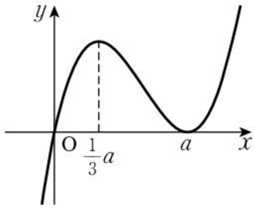
From $f'(x) = 3x^2 - 4ax + a^2 = (3x-a)(x-a) = 0$,

$$x = \frac{1}{3}a,\ a$$

$$f\left(\frac{1}{3}a\right) = \frac{4}{27}a^3,\ f(a) = 0$$

(i) When $a > 0$,

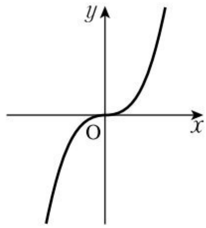
x	$\cdots$	$\frac{1}{3}a$	$\cdots$	a	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	↗	relative maximum	↘	relative minimum	↗



(ii) When $a = 0$,

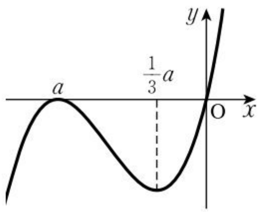
then from $f'(x) = 3x^2 \geqq 0$,

$f(x)$ does not have any relative extreme values.



(iii) When $a < 0$,

x	$\cdots$	a	$\cdots$	$\frac{1}{3}a$	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	↗	relative maximum	↘	relative minimum	↗



From (i), (ii) and (iii):

$$\begin{cases} \textbf{When } a>0\textbf{, the relative maximum value is } \frac{4}{27}a^3\textbf{, at } x=\frac{1}{3}a. \\ \textbf{When } a=0\textbf{, there is no relative maximum value.} \\ \textbf{When } a<0\textbf{, the relative maximum value is 0, at } x=a. \end{cases}$$

L 77a KUMON

Relative Maxima and Minima II

Ex.

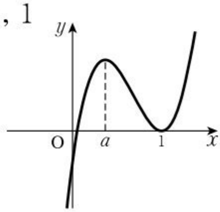
Find the value of a for which $f(x)=2x^3-3(a+1)x^2+6ax-4$ is tangent to the positive x-axis and has a relative minimum at that point of contact.

[Sol] The relative minimum value is 0. ☜ Since the relative minimum is the contact point on the positive x-axis.

From $f'(x)=6x^2-6(a+1)x+6a=6[x^2-(a+1)x+a]$

$=6(x-a)(x-1)=0$, we find $x=a$, 1

(i) When $a<1$,



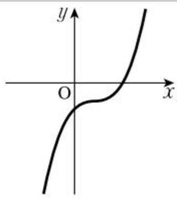
x	$\cdots$	a	$\cdots$	1	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	↗	relative maximum	↘	relative minimum	↗

From $f(1)=2-3a-3+6a-4=0$, ☜ Relative minimum at $x=1$

$a=\boxed{\dfrac{5}{3}}$ is inconsistent with $\underline{a<1}$. ☜ In part (i), the solution must satisfy the underlined condition.

Thus, this is an extraneous solution.

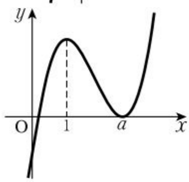
(ii) When $a=1$,



from $f'(x)=6(x-1)^2 \geqq \boxed{0}$,

there are no relative extreme values.

(iii) When $a>1$,



x	$\cdots$	1	$\cdots$	a	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	↗	relative maximum	↘	relative minimum	↗

From $f(a)=2a^3-3(a+1)a^2+6a^2-4$ ☜ Relative minimum at $x=a$

$=-(a+1)(a-2)^2=0$, $a=\boxed{-1}$, $\boxed{2}$

From $\underline{a>1}$, $a=\boxed{2}$ ☜ In part (iii), the solution must satisfy the underlined condition.

From (i), (ii) and (iii), $a=\boxed{2}$

Answers: $\dfrac{5}{3}$, 0 (in order) -1, 2 (in any order) 2, 2 (in order)

L 77b

1. Find the value of a for which $f(x) = -2x^3 + 3(a+2)x^2 - 12ax + 4a^2$ is tangent to the positive x-axis and has a relative maximum at that point of contact.

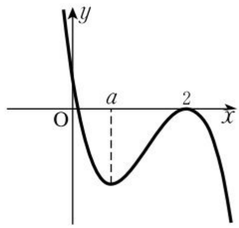
[Sol] The relative maximum value is 0.

From $f'(x) = -6x^2 + 6(a+2)x - 12a = -6[x^2 - (a+2)x + 2a]$

$= -6(x-a)(x-2) = 0$, $x = a$, 2

(i) When $a < 2$,

x	$\cdots$	a	$\cdots$	2	$\cdots$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\searrow$	relative minimum	$\nearrow$	relative maximum	$\searrow$



Since there is a relative maximum at $x = 2$,

from $f(2) = -16 + 12(a+2) - 24a + 4a^2 = 4(a-2)(a-1) = 0$,

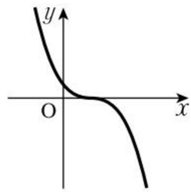
$a = 1$, 2

Since $a < 2$, $a = 1$

(ii) When $a = 2$,

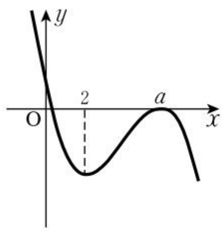
from $f'(x) = -6(x-2)^2 \leqslant 0$,

there are no relative extreme values.



(iii) When $a > 2$,

x	$\cdots$	2	$\cdots$	a	$\cdots$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\searrow$	relative minimum	$\nearrow$	relative maximum	$\searrow$



Since there is a relative maximum at $x = a$,

from $f(a) = -2a^3 + 3(a+2)a^2 - 12a^2 + 4a^2 = a^2(a-2) = 0$,

$a = 0$, 2

Since $a > 2$, $a = 0$, 2 are extraneous solutions.

From (i), (ii) and (iii), $\boldsymbol{a = 1}$

L 78a KUMON

Relative Maxima and Minima II

For each function, create a variation table, and draw the graph. Then, find the relative extreme values.

Ex.

$y=|x(x-4)|$

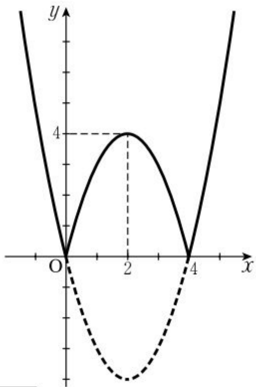
[Sol] When $x(x-4)\geqslant 0$, i.e. when $x\leqslant \boxed{\mathbf{0}}$ or $x\geqslant \boxed{\mathbf{4}}$,

$$y=x(x-4)=x^2-4x$$
$$y'=2x-4=2(x-2) \quad \cdots①$$

When $x(x-4)<0$, i.e. when $\boxed{\mathbf{0}}<x<\boxed{\mathbf{4}}$,

$$y=-x(x-4)=-x^2+4x$$
$$y'=-2x+4=-2(x-2) \quad \cdots②$$

x	$\cdots$	0	$\cdots$	2	$\cdots$	4	$\cdots$
y'	$-$		$+$	0	$-$		$+$
y	↘	0	↗	4	↘	0	↗



Therefore:

The relative minimum value is 0,

at $x=\boxed{\mathbf{0}}$, $\boxed{\mathbf{4}}$.

The relative maximum value is 4, at $x=\boxed{\mathbf{2}}$.

Note: At $x=0$, 4, although $y'=0$ is not true, there are still relative extreme values.

Answers: 0, 4, 0, 4 (in order) 0, 4 (in any order) 2

From ①, as x approaches 0, y' approaches -4. From ②, as x approaches 0, y' approaches 4. From ①, as x approaches 4, y' approaches 4. From ②, as x approaches 4, y' approaches -4. Therefore, although $y'=0$ is not true at $x=0$, 4, there are still relative extreme values at those points. y' is undefined at these points, indicated by crosses as shown.

L 78b

(1) $y=|x^2-1|$

[Sol] $x^2-1=(x+1)(x-1)$

When $x^2-1\geqslant 0$, i.e. when $x\leqslant -1$ or $x\geqslant 1$,

$y=x^2-1$

$y'=2x$

When $x^2-1<0$, i.e. when $-1<x<1$,

$y=-x^2+1$

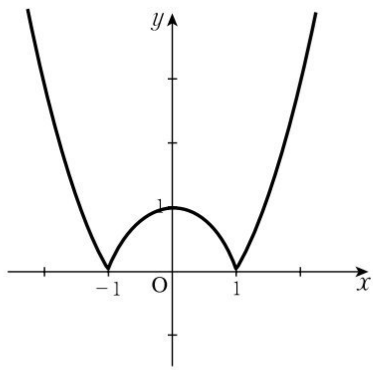
$y'=-2x$

x	$\cdots$	**-1**	$\cdots$	**0**	$\cdots$	**1**	$\cdots$
y'	$-$		$+$	**0**	$-$		$+$
y	↘	**0**	↗	**1**	↘	**0**	↗

Therefore:

The relative minimum value is 0, at $x=-1$, 1.

The relative maximum value is 1, at $x=0$.



L 79a KUMON

Relative Maxima and Minima II

For each function, create a variation table, and draw the graph. Then, find the relative extreme values.

(1) $y = |x(x-3)^2|$

[Sol] When $x(x-3)^2 \geqq 0$, i.e. when $x \geqq \boxed{0}$,

$$y = x(x-3)^2 = \boldsymbol{x^3 - 6x^2 + 9x}$$
$$y' = 3x^2 - 12x + 9 = 3(x-3)(x-1)$$

When $x(x-3)^2 < 0$, i.e. when $x < 0$,

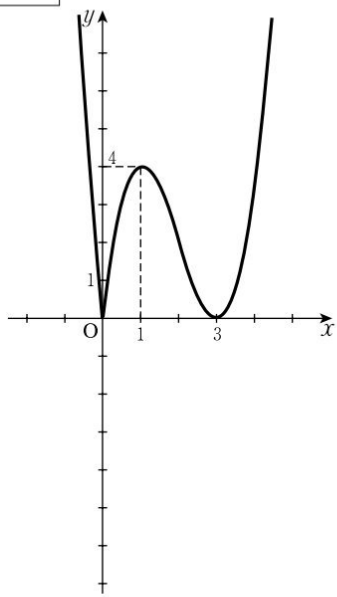
$$y = -x(x-3)^2 = -x^3 + 6x^2 - 9x$$
$$y' = -3x^2 + 12x - 9 = -3(x-3)(x-1)$$

x	$\cdots$	**0**	$\cdots$	**1**	$\cdots$	**3**	$\cdots$
y'	$-$	$\times$	$+$	**0**	$-$	**0**	$+$
y	↘	**0**	↗	**4**	↘	**0**	↗

Therefore:

The relative minimum value is 0, at $x = 0$, 3.

The relative maximum value is 4, at $x = 1$.



L 79b

(2) $y = |x^2(x-3)|$

[Sol] When $x \geqq 3$, The isolated point $(0, 0)$ is not included here.

$$y = x^2(x-3) = x^3 - 3x^2$$

$$y' = 3x^2 - 6x = 3x(x-2)$$

When $x < 3$, The graph is unbroken at the point $(0, 0)$, so it is sensible to include $x = 0$ in this part of the solution.

$$y = -x^2(x-3) = -x^3 + 3x^2$$

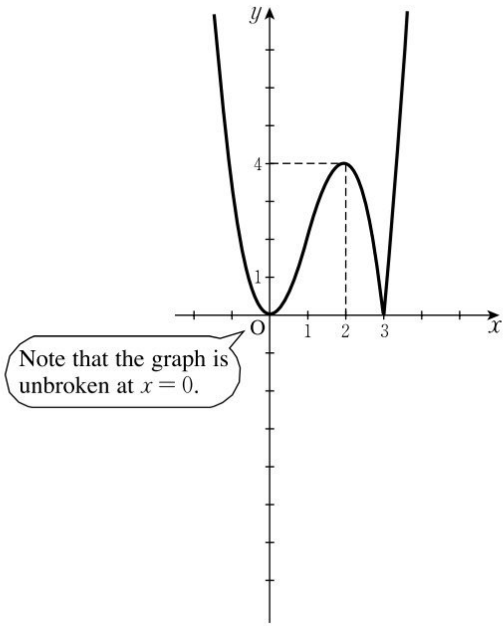
$$y' = -3x^2 + 6x = -3x(x-2)$$

x	$\cdots$	**0**	$\cdots$	**2**	$\cdots$	**3**	$\cdots$
y'	$-$	**0**	$+$	**0**	$-$	$\times$	$+$
y	$\searrow$	**0**	$\nearrow$	**4**	$\searrow$	**0**	$\nearrow$

Therefore:

The relative minimum value is 0, at $x = 0$, 3.

The relative maximum value is 4, at $x = 2$.



L 80a KUMON

Relative Maxima and Minima II

1. Find the relative maximum value of $f(x)=2ax^3-3ax^2-12ax+3$ (where $a\neq 0$).

[Sol] From $f'(x)=6ax^2-6ax-12a$

$=6a(x^2-x-2)$

$=6a(x-2)(x+1)=0,$

$x=-1,\ 2$

(i) When $a>0$,

x	$\cdots$	-1	$\cdots$	2	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	relative maximum	$\searrow$	relative minimum	$\nearrow$

$f(-1)=7a+3$

(ii) When $a<0$,

x	$\cdots$	-1	$\cdots$	2	$\cdots$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\searrow$	relative minimum	$\nearrow$	relative maximum	$\searrow$

$f(2)=-20a+3$

From (i) and (ii):

$$\begin{cases}\textbf{When } a>0\textbf{, the relative maximum value is } 7a+3\textbf{, at } x=-1.\\ \textbf{When } a<0\textbf{, the relative maximum value is } -20a+3\textbf{, at } x=2.\end{cases}$$

L 80b

2. Find the relative minimum value of $f(x)=2x^3-3ax^2+1$.

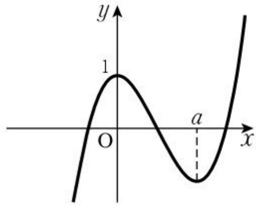
[Sol] From $f'(x)=6x^2-6ax=6x(x-a)=0$,

$x=0,\ a$

$f(0)=1,\ f(a)=-a^3+1$

(i) When $a>0$,

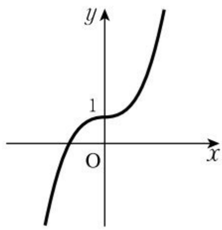
x	$\cdots$	0	$\cdots$	a	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	relative maximum	$\searrow$	relative minimum	$\nearrow$



(ii) When $a=0$,

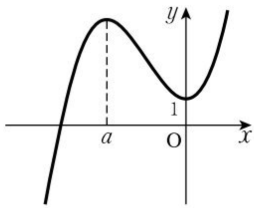
from $f'(x)=6x^2 \geqq 0$,

$f(x)$ does not have any relative extreme values.



(iii) When $a<0$,

x	$\cdots$	a	$\cdots$	0	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	relative maximum	$\searrow$	relative minimum	$\nearrow$



From (i), (ii) and (iii):

$$\begin{cases} \textbf{When } a>0\textbf{, the relative minimum value is } -a^3+1\textbf{, at } x=a. \\ \textbf{When } a=0\textbf{, there is no relative minimum value.} \\ \textbf{When } a<0\textbf{, the relative minimum value is 1, at } x=0. \end{cases}$$

L 81 a KUMON

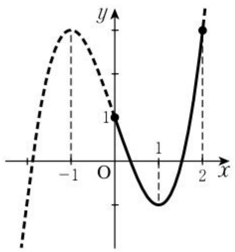
Maxima and Minima I

> To express "any value of x in the domain $a \leqslant x \leqslant b$", we shall use the notation $[a,\ b]$. This represents the closed interval that we are considering.

1. Given the function $f(x)=x^3-3x+1$, find the maximum and minimum values on each interval.

[Sol] $f'(x)=3x^2-3=3(x+1)(x-1)$

x	$\cdots$	-1	$\cdots$	1	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	3	$\searrow$	-1	$\nearrow$



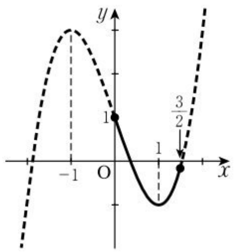
① For $[0,\ 2]$

$f(2)=3$

Therefore:

The maximum value is 3, at $x=2$.

The minimum value is -1, at $x=1$.



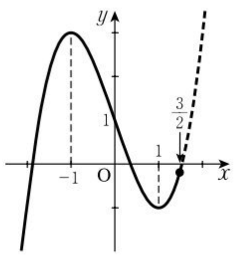
② For $\left[0,\ \dfrac{3}{2}\right]$

$f(0)=1$

Therefore:

The maximum value is $\boxed{1}$, at $x=\boxed{0}$.

The minimum value is $\boxed{-1}$, at $x=\boxed{1}$.



③ For $\left[-3,\ \dfrac{3}{2}\right]$

$f(-3)=-17$

Therefore:

The maximum value is $\boxed{3}$, at $x=\boxed{-1}$.

The minimum value is $\boxed{-17}$, at $x=\boxed{-3}$.

L81b

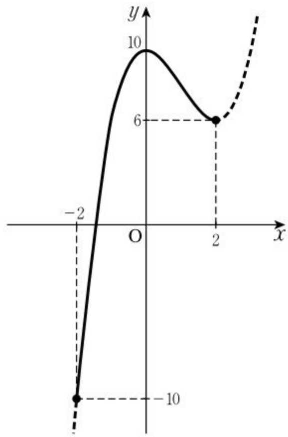
Ex.

Find the maximum and minimum values of the function $y=x^3-3x^2+10$ on the closed interval $[-2,\ 2]$.

[Sol] $y'=3x^2-6x=3x(x-2)$

x	-2	$\cdots$	0	$\cdots$	2
y'	$+$	$+$	0	$-$	0
y	-10	$\nearrow$	10	$\searrow$	6

Include both ends of the interval $x=-2$ and $x=2$.



Therefore:

The maximum value is 10, at $x=0$.

The minimum value is -10, at $x=-2$.

2. Find the maximum and minimum values of the function $y=x^3+3x^2$ on the closed interval $[-3,\ 2]$.

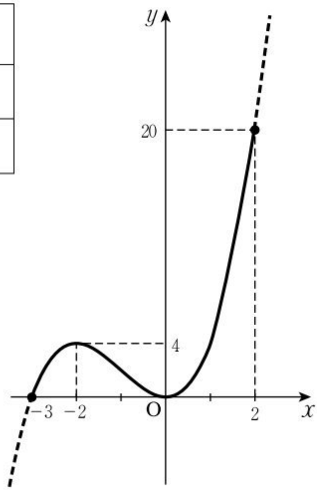
[Sol] $y'=3x^2+6x=3x(x+2)$

x	-3	$\cdots$	-2	$\cdots$	0	$\cdots$	2
y'	$+$	$+$	0	$-$	0	$+$	$+$
y	0	$\nearrow$	4	$\searrow$	0	$\nearrow$	20

Therefore:

The maximum value is 20, at $x=2$.

The minimum value is 0, at $x=-3,\ 0$.



L 82a

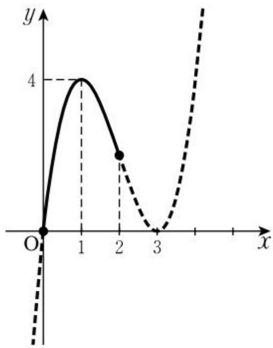
Maxima and Minima I

1. Find the maximum and minimum values of $y=x^3-6x^2+9x$ on the following intervals.

(1) $[0,\ 2]$

[Sol] $y'=3x^2-12x+9$
$\quad\quad =3(x-3)(x-1)$

x	0	$\cdots$	1	$\cdots$	2
y'	+	+	0	−	−
y	0	↗	4	↘	2



Therefore:

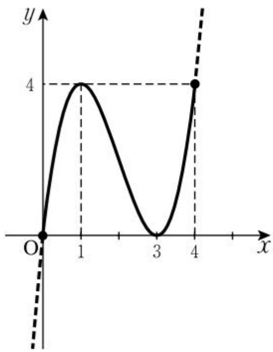
The maximum value is 4, at $x=1$.

The minimum value is 0, at $x=0$.

(2) $[0,\ 4]$

[Sol]

x	0	$\cdots$	1	$\cdots$	3	$\cdots$	4
y'	+	+	0	−	0	+	+
y	0	↗	4	↘	0	↗	4



Therefore:

The maximum value is 4, at $x=1,\ 4$.

The minimum value is 0, at $x=0,\ 3$.

L 82b

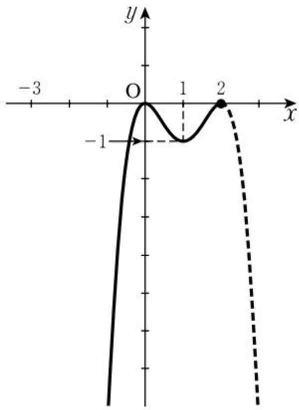
2. Find the maximum and minimum values of $y=-x^4+4x^3-4x^2$ on the following intervals.

(1)　$[-3,\ 2]$

[Sol] $y'=-4x^3+12x^2-8x$

$=-4x(x^2-3x+2)$

$=-4x(x-1)(x-2)$



x	-3	$\cdots$	0	$\cdots$	1	$\cdots$	2
y'	$+$	$+$	0	$-$	0	$+$	0
y	-225	↗	0	↘	-1	↗	0

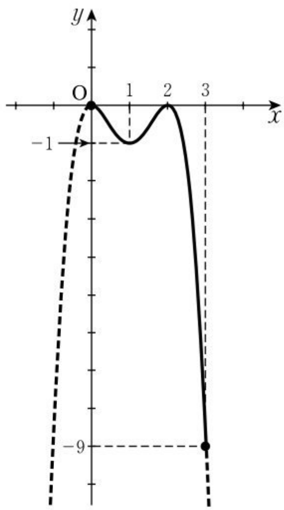
Therefore:

The maximum value is 0, at $x=0,\ 2$.

The minimum value is -225, at $x=-3$.

(2)　$[0,\ 3]$

[Sol]



x	0	$\cdots$	1	$\cdots$	2	$\cdots$	3
y'	0	$-$	0	$+$	0	$-$	$-$
y	0	↘	-1	↗	0	↘	-9

Therefore:

The maximum value is 0, at $x=0,\ 2$.

The minimum value is -9, at $x=3$.

Note: To find a maximum and a minimum value of $y=f(x)$ on an interval $a \leqslant x \leqslant b$:

(i) Create a variation table on the interval $a \leqslant x \leqslant b$.

(ii) Find the relative extreme values and the values of $f(a)$ and $f(b)$ on both ends of the interval, and then compare these.

L 83a KUMON

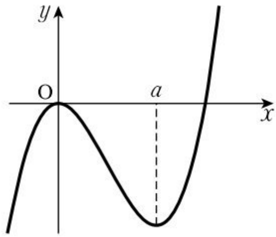
Maxima and Minima I

Ex.

Find the minimum value of $f(x)=x^3-\dfrac{3}{2}ax^2$ on the interval $0\leqslant x\leqslant 2$.
Assume that $a>0$.

[Sol] $f'(x)=3x^2-3ax=3x(x-a)$

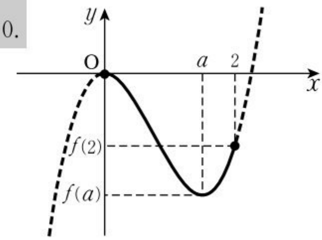
x	$\cdots$	0	$\cdots$	a	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	relative maximum	$\searrow$	relative minimum	$\nearrow$



(i) When $0<a\leqslant 2$,

- Here, the relative minimum value is the minimum value.
- Remember that $a>0$.

the minimum value is $f(a)=\boxed{-\dfrac{a^3}{2}}$.

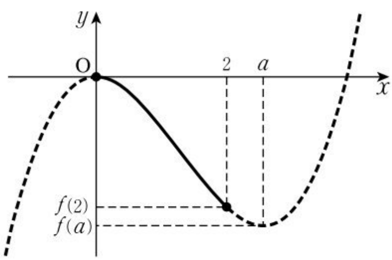


(ii) When $a>2$,

The relative minimum point is not included in the interval.

the minimum value is

$f(2)=\boxed{8-6a}$.



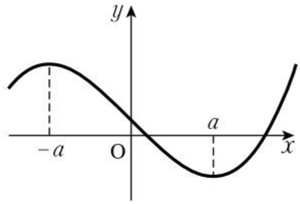
Answers: in order $-\dfrac{a^3}{2}$, $8-6a$

L 83b

1. Find the minimum value of $f(x)=x^3-3a^2x+a^3$ on the interval $0 \leqq x \leqq 2$. Assume that $a>0$.

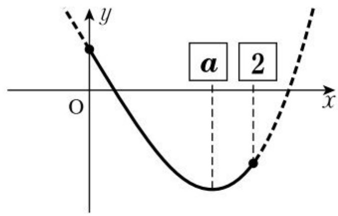
[Sol] $f'(x)=3x^2-3a^2=3(x+a)(x-a)$

x	$\cdots$	$-a$	$\cdots$	a	$\cdots$
$f'(x)$	$+$	**0**	$-$	**0**	$+$
$f(x)$	$\nearrow$	**relative maximum**	$\searrow$	**relative minimum**	$\nearrow$



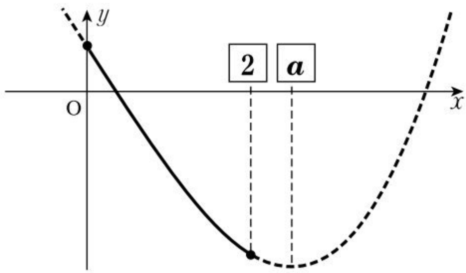
（i） **When $0<a\leqq 2$,**

the minimum value is $\boldsymbol{f(a)=-a^3}$.



（ii） **When $a>2$,**

the minimum value is $\boldsymbol{f(2)=8-6a^2+a^3}$.



When we have to find the minimum value, and there is no relative minimum value in the interval, the minimum value is found either at the right end or the left end of the domain.

L 84a

KUMON

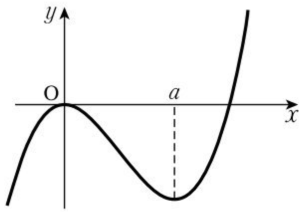
Maxima and Minima I

Ex.

Find the maximum value of $f(x)=x^3-\frac{3}{2}ax^2$ on the interval $0 \leqslant x \leqslant 2$. Assume that $a>0$.

[Sol] $f'(x)=3x^2-3ax=3x(x-a)$

x	$\cdots$	0	$\cdots$	a	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	relative maximum	$\searrow$	relative minimum	$\nearrow$



$f(0)=0$, $f(2)=8-6a$

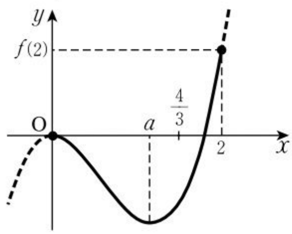
(i) When $f(0) \leqslant f(2)$, ☜ i.e. the maximum is at the right end of the interval.

as $0 \leqslant 8-6a$,

$a \leqslant \frac{4}{3}$

Therefore,

when $0<a \leqslant \frac{4}{3}$, ☜ Remember that $a>0$.

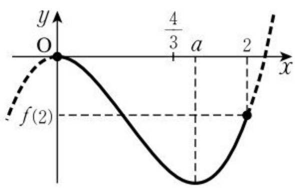


the maximum value is $f(2)=$ $\boxed{\mathbf{8-6a}}$.

(ii) When $f(0)>f(2)$, ☜ i.e. the maximum is at the left end of the interval.

i.e. when $a>\frac{4}{3}$,

the maximum value is $f(0)=$ $\boxed{\mathbf{0}}$.



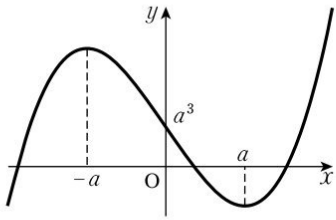
Answers: in order $8-6a$, 0

L 84b

1. Find the maximum value of $f(x)=x^3-3a^2x+a^3$ on the interval $0 \leqq x \leqq 2$. Assume that $a>0$.

[Sol] $f'(x)=3x^2-3a^2=3(x+a)(x-a)$

x	$\cdots$	$-a$	$\cdots$	a	$\cdots$
$f'(x)$	$+$	**0**	$-$	**0**	$+$
$f(x)$	$\nearrow$	**relative maximum**	$\searrow$	**relative minimum**	$\nearrow$



$f(0)=a^3$, $f(2)=8-6a^2+a^3$

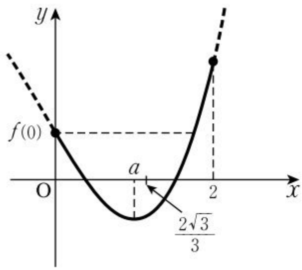
(i) When $f(0) \leqq f(2)$,

as $a^3 \leqq 8-6a^2+a^3$,

$$6a^2-8 \leqq 0$$

$$\left(a+\frac{2\sqrt{3}}{3}\right)\left(a-\frac{2\sqrt{3}}{3}\right) \leqq 0$$

$$-\frac{2\sqrt{3}}{3} \leqq a \leqq \frac{2\sqrt{3}}{3}$$



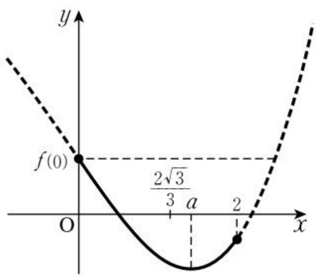
Therefore, **when $0<a \leqq \frac{2\sqrt{3}}{3}$**,

the maximum value is $\boldsymbol{f(2)=8-6a^2+a^3}$.

(ii) When $f(0)>f(2)$,

i.e. **when $a>\frac{2\sqrt{3}}{3}$**,

the maximum value is $\boldsymbol{f(0)=a^3}$.



When we have to find the maximum value, and there is no relative maximum value in the interval, the maximum value is found either at the right end or the left end of the domain.

L 85a KUMON

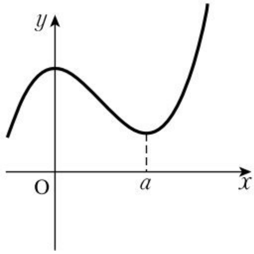
Maxima and Minima I

1. Find the minimum value of $f(x)=2x^3-3ax^2+2a^3$ on the interval $0 \leqq x \leqq 2$. Assume that $a>0$.

[Sol] $f'(x)=6x^2-6ax=6x(x-a)$

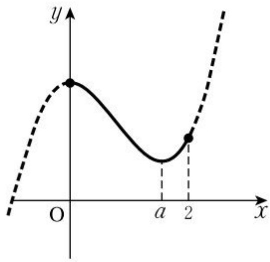
As $a>0$,

x	$\cdots$	0	$\cdots$	a	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	$2a^3$	$\searrow$	a^3	$\nearrow$



(i) When $0<a \leqq 2$,

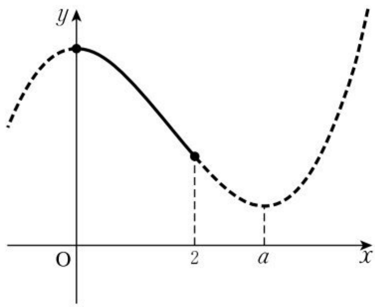
the minimum value is $f(a)=a^3$.



(ii) When $a>2$,

the minimum value is

$f(2)=2a^3-12a+16$.



Answer: When $\boxed{0<a \leqq 2}$, the minimum value is $\boxed{a^3}$, at $x=\boxed{a}$.
When $\boxed{a>2}$, the minimum value is $\boxed{2a^3-12a+16}$, at $x=\boxed{2}$.

L 85b

2. Find the maximum value of the function on side a, $f(x)=2x^3-3ax^2+2a^3$ on the interval $0 \leqslant x \leqslant 2$. Assume that $a>0$.

[Sol] $f(0)=2a^3$

$f(2)=2a^3-12a+16$

(i) When $f(0) \leqslant f(2)$,

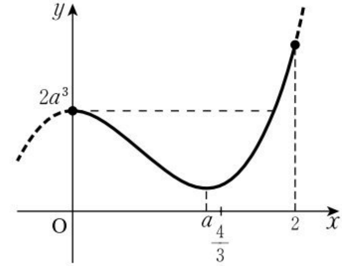
as $2a^3 \leqslant 2a^3-12a+16$,

$12a \leqslant 16$

$a \leqslant \dfrac{4}{3}$

Therefore, when $0<a \leqslant \dfrac{4}{3}$,

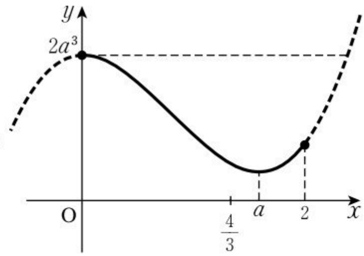
the maximum value is $f(2)=2a^3-12a+16$.



(ii) When $f(0)>f(2)$,

i.e. when $a>\dfrac{4}{3}$,

the maximum value is $f(0)=2a^3$.



Answer:

When $\boxed{\mathbf{0<\boldsymbol{a}\leqslant \frac{4}{3}}}$,

the maximum value is $\boxed{\mathbf{2\boldsymbol{a}^3-12\boldsymbol{a}+16}}$, at $x=\boxed{\mathbf{2}}$.

When $\boxed{\mathbf{\boldsymbol{a}>\frac{4}{3}}}$,

the maximum value is $\boxed{\mathbf{2\boldsymbol{a}^3}}$, at $x=\boxed{\mathbf{0}}$.

L 86a KUMON

Maxima and Minima I

Ex.

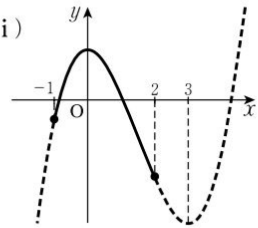
Find the minimum value of $f(x)=2ax^3-9ax^2+5a$ on the interval $-1 \leqq x \leqq 2$.

[Sol] $f'(x)=6ax^2-18ax=6ax(x-3)$

(i) When $a>0$,

x	-1	$\cdots$	0	$\cdots$	2
$f'(x)$	$+$	$+$	0	$-$	$-$
$f(x)$	$-6a$	$\nearrow$	$5a$	$\searrow$	$-15a$

(i)



As $f(-1)-f(2)=9a>0$, ☜ Since $a>0$.

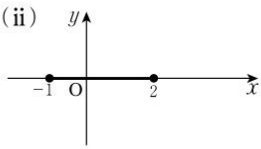
$f(-1)>f(2)$

Therefore, the minimum value is $f(\boxed{\ 2\ })=\boxed{-15a}$.

(ii) When $a=0$, $f(x)=0$

Therefore, the minimum value is 0.

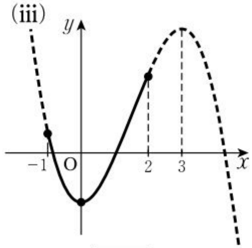
(ii)



(iii) When $a<0$,

x	-1	$\cdots$	0	$\cdots$	2
$f'(x)$	$-$	$-$	$\mathbf{0}$	$+$	$+$
$f(x)$	$\mathbf{-6a}$	$\searrow$	$\mathbf{5a}$	$\nearrow$	$\mathbf{-15a}$

(iii)



Therefore, the minimum value is $f(\boxed{\ 0\ })=\boxed{5a}$.

From (i), (ii) and (iii):

$$\begin{cases} \text{When } a>0, \text{ the minimum value is } \boxed{-15a}, \text{ at } x=\boxed{2}. \\ \text{When } a=0, \text{ the minimum value is } 0. \\ \text{When } a<0, \text{ the minimum value is } \boxed{5a}, \text{ at } x=\boxed{0}. \end{cases}$$

Answers: in order 2, $-15a$,

x	-1	$\cdots$	0	$\cdots$	2
$f'(x)$	$-$	$-$	0	$+$	$+$
$f(x)$	$-6a$	$\searrow$	$5a$	$\nearrow$	$-15a$

, 0, $5a$, $-15a$, 2, $5a$, 0

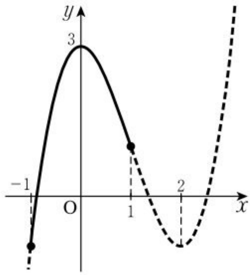
L 86b

1. Find the minimum value of $f(x)=ax^3-3ax^2+3$ on the interval $-1\leqslant x\leqslant 1$.

[Sol] $f'(x)=3ax^2-6ax=3ax(x-2)$

(i) When $a>0$,

x	-1	$\cdots$	0	$\cdots$	1
$f'(x)$	$+$	$+$	0	$-$	$-$
$f(x)$	$-4a+3$	$\nearrow$	3	$\searrow$	$-2a+3$



As $f(-1)-f(1)=-2a<0$,

$f(-1)<f(1)$

Therefore, the minimum value is $f(-1)=-4a+3$.

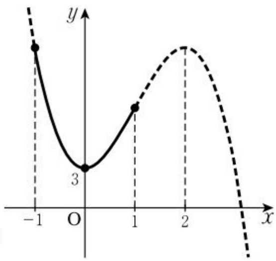
(ii) When $a=0$, $f(x)=3$

Therefore, the minimum value is 3.

(iii) When $a<0$,

x	-1	$\cdots$	0	$\cdots$	1
$f'(x)$	$-$	$-$	0	$+$	$+$
$f(x)$	$-4a+3$	$\searrow$	3	$\nearrow$	$-2a+3$

Therefore, the minimum value is $f(0)=3$.



From (i), (ii) and (iii):

$$\begin{cases} \textbf{When } a>0\textbf{, the minimum value is } -4a+3\textbf{, at } x=-1. \\ \textbf{When } a=0\textbf{, the minimum value is } 3. \\ \textbf{When } a<0\textbf{, the minimum value is } 3\textbf{, at } x=0. \end{cases}$$

L 87a KUMON

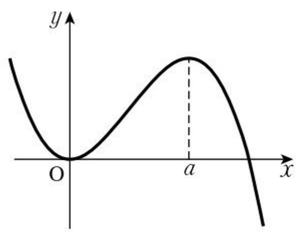
Maxima and Minima I

1. Find the minimum value of $f(x)=-2ax^3+3a^2x^2$ on the interval $0 \leqslant x \leqslant 2$. Assume that $a>0$.

[Sol] $f'(x)=-6ax^2+6a^2x=-6ax(x-a)$

As $a>0$,

x	$\cdots$	0	$\cdots$	a	$\cdots$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\searrow$	0	$\nearrow$	a^4	$\searrow$



$f(0)=0,\ f(2)=-16a+12a^2$

(i) When $f(0) \geqslant f(2)$,

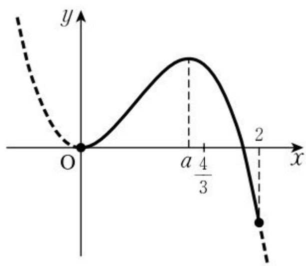
as $0 \geqslant -16a+12a^2$,

$4a(3a-4) \leqslant 0$

Therefore, when $0<a \leqslant \dfrac{4}{3}$,

the minimum value is

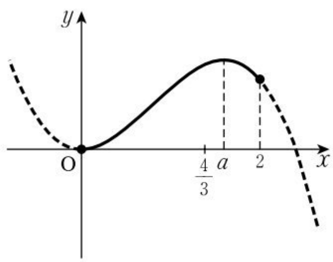
$f(2)=-16a+12a^2$



(ii) When $f(0)<f(2)$,

i.e. when $a>\dfrac{4}{3}$,

the minimum value is $f(0)=0$.



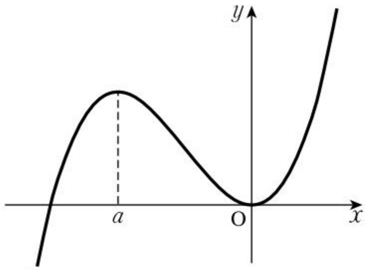
From (i) and (ii):

$$\begin{cases} \textbf{When } 0<a \leqslant \dfrac{4}{3}\textbf{, the minimum value is } -16a+12a^2\textbf{, at } x=2. \\ \textbf{When } a>\dfrac{4}{3}\textbf{, the minimum value is } 0\textbf{, at } x=0. \end{cases}$$

2. Find the minimum value of the function on side a, $f(x)=-2ax^3+3a^2x^2$, on the interval $0 \leqslant x \leqslant 2$ when $a<0$.

[Sol] As $a<0$,

x	$\cdots$	a	$\cdots$	0	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	a^4	$\searrow$	0	$\nearrow$



Therefore, **the minimum value is 0, at $x=0$.**

Let's try this!

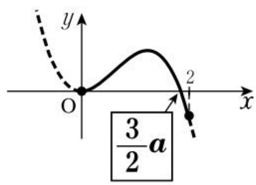
On side a, we studied the function $f(x)$ based on the values of x in the given domain.

Another way to find the minimum values:

Let $f(x)=f(0)$ and solve for x.

From $-2ax^3+3a^2x^2=0$,

$x=0,\ \boxed{\frac{3}{2}a}$

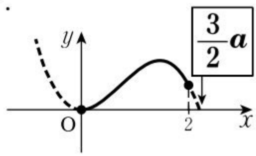


(i) When $\boxed{\frac{3}{2}a} \leqslant 2$, i.e. when $0<a\leqslant \frac{4}{3}$,

the minimum value is $f(2)=-16a+12a^2$.

(ii) When $\boxed{\frac{3}{2}a} > 2$, i.e. when $a>\frac{4}{3}$,

the minimum value is $f(0)=0$.



Answers: All the answers are the same, $\frac{3}{2}a$.

L 88a KUMON

Maxima and Minima I

Ex.

Find the maximum value of $f(x) = x^3 - 6ax^2 + 9a^2x$ on the interval $0 \leqslant x \leqslant 2$. Assume that $a > 0$.

[Sol] $f'(x) = 3x^2 - 12ax + 9a^2 = 3(x^2 - 4ax + 3a^2)$

$= 3(x - 3a)(x - a)$

Since $a > 0$,

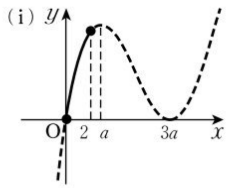
x	0	$\cdots$	a	$\cdots$	$3a$	$\cdots$
$f'(x)$	$+$	$+$	0	$-$	0	$+$
$f(x)$	0	$\nearrow$	$4a^3$	$\searrow$	0	$\nearrow$

Find the value of $x > 3a$ where $f(x) = 4a^3$.

$x^3 - 6ax^2 + 9a^2x = 4a^3$

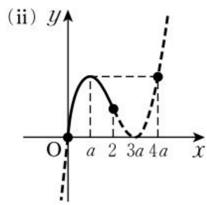
$(x - a)^2(x - 4a) = 0$

$x = 4a$



(i) When $a > 2$,

the maximum value is $f(\ \mathbf{2}\ ) = 18a^2 - 24a + 8$.

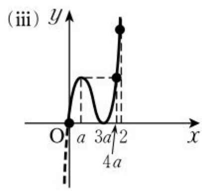


(ii) When $a \leqslant 2 \leqslant 4a$, i.e. when $\dfrac{1}{2} \leqslant a \leqslant 2$,

the maximum value is $f(\ \boldsymbol{a}\ ) = 4a^3$.

(iii) When $4a < 2$, i.e. when $0 < a < \dfrac{1}{2}$,

the maximum value is $f(\ \mathbf{2}\ ) = 18a^2 - 24a + 8$.



From (i), (ii) and (iii):

$$\begin{cases} \text{When } 0 < a < \dfrac{1}{2} \text{ and when } a > 2, \text{ the maximum} \\ \text{value is } 18a^2 - 24a + 8, \text{ at } x = \boxed{\mathbf{2}}. \\ \text{When } \dfrac{1}{2} \leqslant a \leqslant 2, \text{ the maximum value is } 4a^3, \text{ at } x = \boxed{\boldsymbol{a}}. \end{cases}$$

Answers: in order 2, a, 2, 2, a

L 88b

1. Find the maximum value of $f(x)=x^3-6ax^2+9a^2x-2a^3$ on the interval $0\leqslant x\leqslant 2$. Assume that $a>0$.

[Sol] $f'(x)=3x^2-12ax+9a^2=3(x^2-4ax+3a^2)$

$=3(x-3a)(x-a)$

Since $a>0$,

x	0	$\cdots$	a	$\cdots$	$3a$	$\cdots$
$f'(x)$	$+$	$+$	0	$-$	0	$+$
$f(x)$	$-2a^3$	$\nearrow$	$2a^3$	$\searrow$	$-2a^3$	$\nearrow$

Find the value of $x>3a$ where $f(x)=2a^3$.

$x^3-6ax^2+9a^2x-2a^3=2a^3$

$(x-a)^2(x-4a)=0$

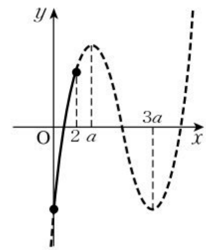
$x=4a$

(i) When $a>2$,

the maximum value is

$f(2)=-2a^3+18a^2-24a+8.$

(i)



(ii) When $a\leqslant 2\leqslant 4a$, i.e. when $\dfrac{1}{2}\leqslant a\leqslant 2$,

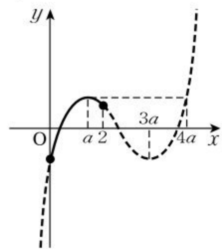
the maximum value is $f(a)=2a^3$.

(iii) When $4a<2$, i.e. when $0<a<\dfrac{1}{2}$,

the maximum value is

$f(2)=-2a^3+18a^2-24a+8.$

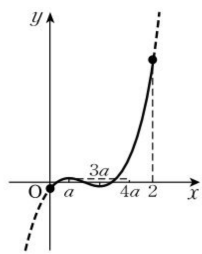
(ii)



From (i), (ii) and (iii):

$$\begin{cases} \textbf{When } 0<a<\dfrac{1}{2} \textbf{ and when } a>2\textbf{, the maximum value is } -2a^3+18a^2-24a+8\textbf{, at } x=2. \\ \textbf{When } \dfrac{1}{2}\leqslant a\leqslant 2\textbf{, the maximum value is } 2a^3\textbf{, at } x=a. \end{cases}$$

(iii)



L 89a KUMON

Maxima and Minima I

Ex.

Find the values of a and b for which $f(x)=ax^3-6ax^2+b$ has a maximum value of 3 and a minimum value of -29 on the interval $-1\leqslant x\leqslant 2$. Assume that $a>0$.

[Sol] $f'(x)=3ax^2-12ax=3ax(x-4)$

$f(-1)=-7a+b$, $f(2)=-16a+b$

Since $a>0$,

x	-1	$\cdots$	0	$\cdots$	2
$f'(x)$	$+$	$+$	0	$-$	$-$
$f(x)$	$-7a+b$	$\nearrow$	b	$\searrow$	$-16a+b$

Since the maximum is at $x=0$,

$f(0)=b=\boxed{\mathbf{3}}$ $\cdots$①

As $f(-1)-f(2)=9a>0$,

$f(-1)>f(2)$

Therefore, since the minimum is at $x=2$,

$f(2)=-16a+b=\boxed{\mathbf{-29}}$ $\cdots$②

From ① and ②,

$a=\boxed{\mathbf{2}}$, $b=\boxed{\mathbf{3}}$

Answers: in order 3, -29, 2, 3

L 89b

1. Find the values of a and b for which $f(x)=ax^3+3ax^2+b$ has a maximum value of 10 and a minimum value of -8 on the interval $-1 \leqslant x \leqslant 2$. Assume that $a<0$.

[Sol] $f'(x)=3ax^2+6ax=3ax(x+2)$

$f(-1)=2a+b$, $f(2)=20a+b$

Since $a<0$,

x	-1	$\cdots$	0	$\cdots$	2
$f'(x)$	$+$	$+$	0	$-$	$-$
$f(x)$	$2a+b$	$\nearrow$	b	$\searrow$	$20a+b$

Since the maximum is at $x=0$,

$f(0)=b=10 \qquad \cdots①$

As $f(-1)-f(2)=-18a>0$,

$f(-1)>f(2)$

Therefore, since the minimum is at $x=2$,

$f(2)=20a+b=-8 \qquad \cdots②$

From ① and ②,

$$\boldsymbol{a=-\frac{9}{10},\ b=10}$$

L 90a KUMON

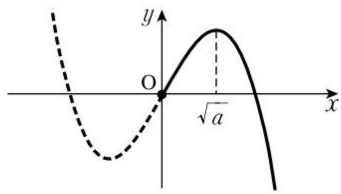
Maxima and Minima I

1. Find the minimum value of $f(x)=-x^3+3ax$ on the interval $0 \leqslant x \leqslant 2$. Assume that $a>0$.

[Sol] $f'(x)=-3x^2+3a=-3(x^2-a)=-3(x+\sqrt{a})(x-\sqrt{a})$

Since $a>0$,

x	$\cdots$	$-\sqrt{a}$	$\cdots$	$\sqrt{a}$	$\cdots$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	↘	relative minimum	↗	relative maximum	↘

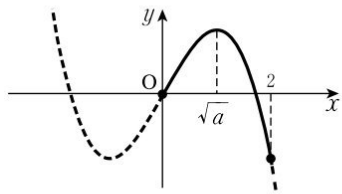


$f(0)=0$, $f(2)=-8+6a$

(i) When $f(0) \geqslant f(2)$,

as $0 \geqslant -8+6a$,

$a \leqslant \dfrac{4}{3}$



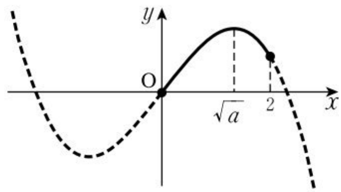
Therefore, when $0<a \leqslant \dfrac{4}{3}$,

the minimum value is $f(2)=-8+6a$.

(ii) When $f(0)<f(2)$,

i.e. when $a>\dfrac{4}{3}$,

the minimum value is $f(0)=0$.



From (i) and (ii):

$$\begin{cases} \textbf{When } 0<a \leqslant \dfrac{4}{3}\textbf{, the minimum value is } -8+6a\textbf{, at } x=2. \\ \textbf{When } a>\dfrac{4}{3}\textbf{, the minimum value is 0, at } x=0. \end{cases}$$

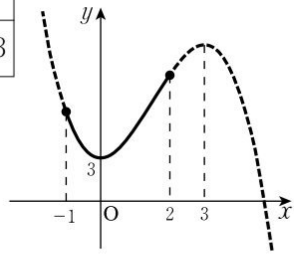
L 90b

2. Find the maximum value of $f(x)=2ax^3-9ax^2+3$ on the interval $-1 \leqq x \leqq 2$. Assume that $a<0$.

[Sol] $f'(x)=6ax^2-18ax=6ax(x-3)$

Since $a<0$,

x	-1	$\cdots$	0	$\cdots$	2
$f'(x)$	$-$	$-$	0	$+$	$+$
$f(x)$	$-11a+3$	↘	relative minimum	↗	$-20a+3$



As $f(2)-f(-1)=-9a>0$,

$f(2)>f(-1)$

Therefore,

the maximum value is $-20a+3$, at $x=2$.

Let's try this!

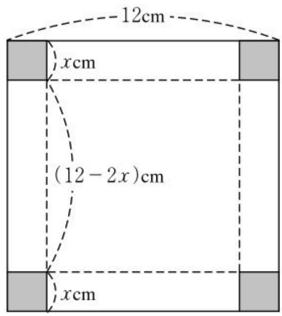
As shown on the figure below, we will make an open box by cutting off 4 squares from the square with length 12 cm. What size should the squares be in order to obtain a box with the largest possible volume? Find the length of the sides of the squares.

$f(x)=(12-2x)^2x=4x^3-48x^2+144x$ ☜ length × width × height

$f'(x)=12x^2-96x+144=12(x-6)(x-2)$

In order to create a box, $0<12-2x<12$

This means $0<x<6$.



x	0	$\cdots$	$\boxed{2}$	$\cdots$	6
$f'(x)$	/	$+$	0	$-$	/
$f(x)$	/	↗	relative maximum	↘	/

Ans. Squares with side length $\boxed{2}$ cm.

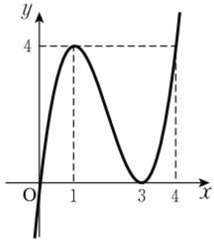
Answers: All the answers are the same, 2.

L91a KUMON

Maxima and Minima II

Ex.

Find the maximum value of $f(x) = x^3 - 6x^2 + 9x$ on the interval $0 \leqq x \leqq a$, $(a > 0)$.



[Sol] $f'(x) = 3x^2 - 12x + 9$

$= 3(x^2 - 4x + 3) = 3(x-1)(x-3)$

x	$\cdots$	1	$\cdots$	3	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	4	$\searrow$	0	$\nearrow$

Find the value of $x > 1$ where $f(x) = 4$.

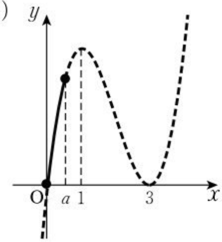
$x^3 - 6x^2 + 9x = 4$ ☜ Given $x > 1$, find the value of x at which $f(x)$ equals the relative maximum value.

$x^3 - 6x^2 + 9x - 4 = 0$

$(x-1)^2(x-4) = 0$

$x = 4$

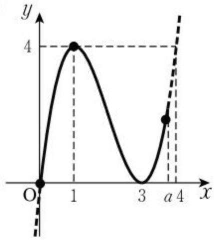
(i)



(i) When $0 < a < 1$, ☜ There is no relative maximum value in this interval.

the maximum value is $f(a) = a^3 - 6a^2 + 9a$.

(ii)

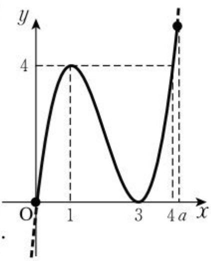


(ii) When $1 \leqq a \leqq \boxed{4}$, ☜ There is a relative maximum value in this interval.

the maximum value is $f(1) = \boxed{4}$.

(**Note:** When $a = 4$, then $f(1) = f(4) = 4$.)

(iii)



(iii) When $a > \boxed{4}$, ☜ The value at the right end of the interval is greater than the relative maximum value.

the maximum value is $f(\ \boldsymbol{a}\ ) = \boxed{\boldsymbol{a}^3 - 6\boldsymbol{a}^2 + 9\boldsymbol{a}}$.

Answers: in order 4, 4, 4, a, $a^3 - 6a^2 + 9a$

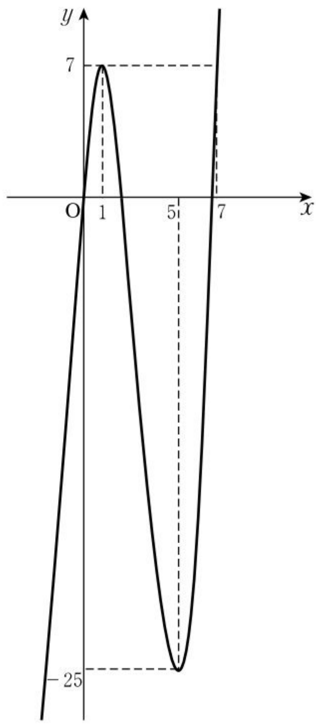
L91b

1. Find the maximum value of $f(x)=x^3-9x^2+15x$ on the interval $0 \leqslant x \leqslant a$, $(a>0)$.

[Sol] $f'(x)=3x^2-18x+15$

$\quad = 3(x^2-6x+5)=3(x-5)(x-1)$

x	$\cdots$	1	$\cdots$	5	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	7	$\searrow$	-25	$\nearrow$



Find the value of $x>1$ where $f(x)=7$.

$x^3-9x^2+15x=7$

$x^3-9x^2+15x-7=0$

$(x-1)^2(x-7)=0$

$x=7$

(i) **When $0<a<1$,**

the maximum value is

$\boldsymbol{f(a)=a^3-9a^2+15a}$.

(ii) **When $1 \leqslant a \leqslant 7$,**

the maximum value is

$\boldsymbol{f(1)=7}$.

(iii) **When $a>7$,**

the maximum value is

$\boldsymbol{f(a)=a^3-9a^2+15a}$.

L 92a KUMON

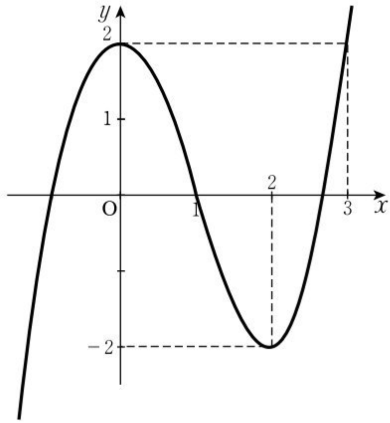
Maxima and Minima II

1. Find the maximum value of $f(x)=x^3-3x^2+2$ on the interval $0 \leqslant x \leqslant a$, $(a>0)$.

[Sol] $f'(x)=3x^2-6x$

$=3x(x-2)$

x	$\cdots$	0	$\cdots$	2	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	2	$\searrow$	-2	$\nearrow$



Find the value of $x>0$ where $f(x)=2$.

$x^3-3x^2+2=2$

$x^3-3x^2=x^2(x-3)=0$

$x=3$

(i) **When $0<a\leqslant 3$,**

the maximum value is $\boldsymbol{f(0)=2}$.

(ii) **When $a>3$,**

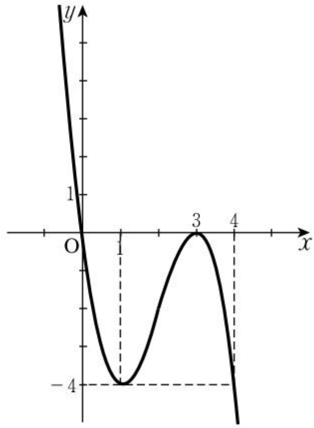
the maximum value is $\boldsymbol{f(a)=a^3-3a^2+2}$.

L 92b

2. Find the minimum value of $f(x) = -x^3 + 6x^2 - 9x$ on the interval $0 \leqslant x \leqslant a$, $(a > 0)$.

[Sol] $f'(x) = -3x^2 + 12x - 9$
$= -3(x-3)(x-1)$

x	$\cdots$	1	$\cdots$	3	$\cdots$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\searrow$	-4	$\nearrow$	0	$\searrow$



Find the value of $x > 1$ where $f(x) = -4$.

$-x^3 + 6x^2 - 9x = -4$

$x^3 - 6x^2 + 9x - 4 = 0$

$(x-1)^2(x-4) = 0$

$x = 4$

(i) **When $0 < a < 1$,**
the minimum value is $\boldsymbol{f(a) = -a^3 + 6a^2 - 9a}$.

(ii) **When $1 \leqslant a \leqslant 4$,**
the minimum value is $\boldsymbol{f(1) = -4}$.

(iii) **When $a > 4$,**
the minimum value is $\boldsymbol{f(a) = -a^3 + 6a^2 - 9a}$.

L 93a KUMON

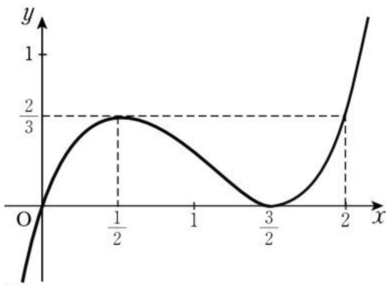
Maxima and Minima II

1. Find the maximum value of $f(x)=\dfrac{4}{3}x^3-4x^2+3x$ on the interval $0 \leqslant x \leqslant a,\ (a>0)$.

[Sol] $f'(x)=4x^2-8x+3$

$=(2x-3)(2x-1)$

x	$\cdots$	$\frac{1}{2}$	$\cdots$	$\frac{3}{2}$	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	$\frac{2}{3}$	$\searrow$	0	$\nearrow$



Find the value of $x>\dfrac{1}{2}$ where $f(x)=\dfrac{2}{3}$.

$\dfrac{4}{3}x^3-4x^2+3x=\dfrac{2}{3}$

$4x^3-12x^2+9x-2=0$

$(2x-1)^2(x-2)=0$

$x=2$

(i) **When $0<a<\dfrac{1}{2}$,**

the maximum value is $\boldsymbol{f(a)=\dfrac{4}{3}a^3-4a^2+3a}$.

(ii) **When $\dfrac{1}{2} \leqslant a \leqslant 2$,**

the maximum value is $\boldsymbol{f\left(\dfrac{1}{2}\right)=\dfrac{2}{3}}$.

(iii) **When $a>2$,**

the maximum value is $\boldsymbol{f(a)=\dfrac{4}{3}a^3-4a^2+3a}$.

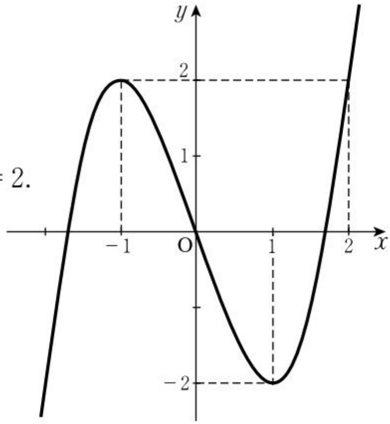
L 93b

2.* Find the maximum value of $f(x)=x^3-3x$ on the interval $-a \leqq x \leqq a$, $(a>0)$.

[Sol] $f'(x)=3x^2-3$

$=3(x+1)(x-1)$

x	$\cdots$	-1	$\cdots$	1	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	2	$\searrow$	-2	$\nearrow$



Find the value of $x>-1$ where $f(x)=2$.

$x^3-3x=2$

$x^3-3x-2=0$

$(x+1)^2(x-2)=0$

$x=2$

(i) **When $0<a<1$,**

the maximum value is $\boldsymbol{f(-a)=-a^3+3a}$.

(ii) **When $1\leqq a\leqq 2$,**

the maximum value is $\boldsymbol{f(-1)=2}$.

(iii) **When $a>2$,**

the maximum value is $\boldsymbol{f(a)=a^3-3a}$.

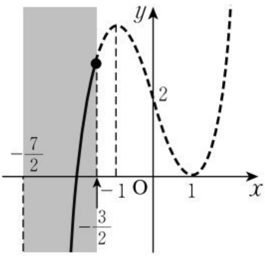
L 94a KUMON

Maxima and Minima II

1. For each given value of a, find the location of the maximum value of $f(x)=x^3-3x+2$ in the domain $a \leqslant x \leqslant a+2$.

Ex.

When $a=-\dfrac{7}{2}$



[Sol] The domain becomes:

$-\dfrac{7}{2} \leqslant x \leqslant -\dfrac{3}{2}$ ☜ $a=-\dfrac{7}{2},\ a+2=-\dfrac{3}{2}$

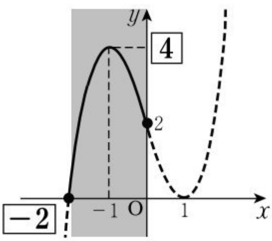
From the graph,

the maximum value is $f\left(-\dfrac{3}{2}\right)=\dfrac{25}{8}$.

In this case, the maximum value is located at:

{ The left end of the domain / The relative maximum value / The right end of the domain }

(1) When $a=-2$



[Sol] The domain becomes:

$\boldsymbol{-2 \leqslant x \leqslant 0}$

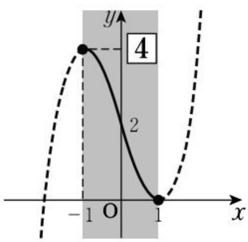
From the graph,

the maximum value is $\boldsymbol{f(-1)=4}$.

In this case, the maximum value is located at:

{ The left end of the domain / The relative maximum value / The right end of the domain }

(2) When $a=-1$



[Sol] The domain becomes:

$\boldsymbol{-1 \leqslant x \leqslant 1}$

From the graph,

the maximum value is $\boldsymbol{f(-1)=4}$.

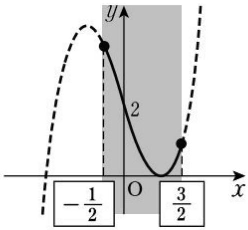
In this case, the maximum value is located at:

{ The left end of the domain / The relative maximum value / The right end of the domain }

Note: In this case both of the above apply.

L 94b

(3) When $a=-\frac{1}{2}$ ☞ Compare both sides of the interval.



[Sol] The domain becomes:

$$-\frac{1}{2} \leqslant x \leqslant \frac{3}{2}$$

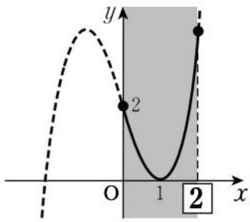
$f\left(-\frac{1}{2}\right)=\frac{27}{8}$ $\quad f\left(\frac{3}{2}\right)=\frac{7}{8}$

The maximum value is $f\left(-\frac{1}{2}\right)=\frac{27}{8}$.

In this case, the maximum value is located at:

{ The left end of the domain / The relative maximum value / The right end of the domain }

(4) When $a=0$



[Sol] The domain becomes:

$$0 \leqslant x \leqslant 2$$

$f(0)=2 \quad f(2)=4$

The maximum value is $f(2)=4$.

In this case, the maximum value is located at:

{ The left end of the domain / The relative maximum value / The right end of the domain }

2. Given the function $f(x)=x^3-3x+2$ on the domain $a \leqslant x \leqslant a+2$, find the value of a at which the two ends of the domain give the same value, equal to the maximum value of the interval.

[Sol] $f(a)=a^3-3a+2$

$f(a+2)=(a+2)^3-3(a+2)+2=a^3+6a^2+9a+4$

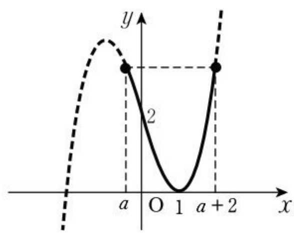
If $f(a)=f(a+2)$,

$a^3-3a+2=a^3+6a^2+9a+4$

$6a^2+12a+2=0$

$3a^2+6a+1=0$

$a=\frac{-3 \pm \sqrt{6}}{3}$



As $a<1<a+2$, then $-1<a<1$

Therefore, $a=\frac{-3+\sqrt{6}}{3}$

L 95a KUMON

Maxima and Minima II

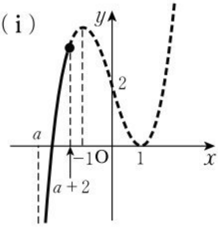
Ex.

Find the maximum value of $f(x)=x^3-3x+2$ in the domain $a\leqslant x\leqslant a+2$.

[Sol] $f'(x)=3x^2-3=3(x+1)(x-1)$

$f(x)$ has a relative maximum at $x=-1$,

and a relative minimum at $x=1$.

(i) 

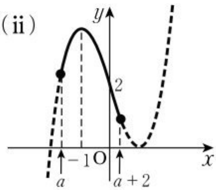
(i) When $a+2<-1$, ☜ The domain is to the left of the relative maximum value.

i.e. when $a<-3$,

the maximum value is

$f(\ \boldsymbol{a+2}\ )=(a+2)^3-3(a+2)+2$

$=a^3+6a^2+9a+4$

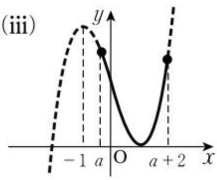
(ii) 

(ii) When $a\leqslant-1\leqslant a+2$, ☜ The relative maximum value is within the domain.

i.e. when $-3\leqslant a\leqslant-1$,

the maximum value is $f(\ \boldsymbol{-1}\ )=\boxed{\mathbf{4}}$.

(iii) If $a>-1$, first consider the case when both ends of the domain $a\leqslant x\leqslant a+2$ give the same value:

(iii) 

$f(a)=f(a+2)$

$a^3-3a+2=a^3+6a^2+9a+4$

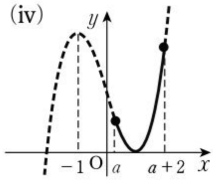
$2(3a^2+6a+1)=0$

Since $a>-1$, $a=\dfrac{-3+\sqrt{6}}{3}$

Therefore,

when $-1<a<\dfrac{-3+\sqrt{6}}{3}$, ☜ The maximum value is at the left end of the domain.

the maximum value is $f(\ \boldsymbol{a}\ )=\boxed{\boldsymbol{a^3-3a+2}}$.

(iv) 

(iv) When $a\geqslant\boxed{\dfrac{\mathbf{-3+\sqrt{6}}}{\mathbf{3}}}$, ☜ The maximum value is at the right end of the domain.

the maximum value is $f(a+2)=a^3+6a^2+9a+4$.

Answers: in order $a+2$, -1, 4, a, a^3-3a+2, $\dfrac{-3+\sqrt{6}}{3}$

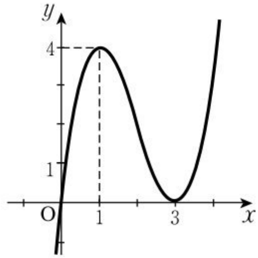
L 95b

1. Find the maximum value of $f(x)=x^3-6x^2+9x$ in the domain $a \leqslant x \leqslant a+2$.

[Sol] $f'(x)=3x^2-12x+9$

$=3(x^2-4x+3)=3(x-3)(x-1)$

$f(x)$ has a relative maximum at $x=1$, and a relative minimum at $x=3$.



(i) When $a+2<1$, i.e. **when $\boldsymbol{a<-1}$,**

the maximum value is

$\boldsymbol{f(a+2)}=(a+2)^3-6(a+2)^2+9(a+2)$

$\boldsymbol{=a^3-3a+2}$

(ii) When $a \leqslant 1 \leqslant a+2$, i.e. **when $\boldsymbol{-1 \leqslant a \leqslant 1}$,**

the maximum value is $\boldsymbol{f(1)=4}$.

(iii) If $a>1$, first consider the case when both ends of the domain $a \leqslant x \leqslant a+2$ give the same value:

$f(a)=f(a+2)$

$a^3-6a^2+9a=a^3-3a+2$

$2(3a^2-6a+1)=0$

Since $a>1$, $a=\dfrac{3+\sqrt{6}}{3}$

Therefore,

when $\boldsymbol{1<a<\dfrac{3+\sqrt{6}}{3}}$,

the maximum value is $\boldsymbol{f(a)=a^3-6a^2+9a}$.

(iv) **When $\boldsymbol{a \geqslant \dfrac{3+\sqrt{6}}{3}}$,**

the maximum value is $\boldsymbol{f(a+2)=a^3-3a+2}$.

L 96a KUMON

Maxima and Minima II

1. Find the minimum value of $f(x)=x^3-6x^2+9x$ in the domain $a\leqslant x\leqslant a+2$.

[Sol] $f'(x)=3x^2-12x+9$

$=3(x^2-4x+3)=3(x-3)(x-1)$

$f(x)$ has a relative maximum at $x=1$, and a relative minimum at $x=3$.

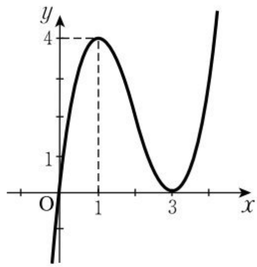
(i) If $a<1$, first consider when both ends of the domain $a\leqslant x\leqslant a+2$ give the same value:

$f(a)=f(a+2)$

$a^3-6a^2+9a=a^3-3a+2$

$2(3a^2-6a+1)=0$

Since $a<1$, $a=\dfrac{3-\sqrt{6}}{3}$



Therefore,

when $\boldsymbol{a<\dfrac{3-\sqrt{6}}{3}}$,

the minimum value is $\boldsymbol{f(a)=a^3-6a^2+9a}$.

(ii) When $\dfrac{3-\sqrt{6}}{3}\leqslant a$ and $a+2<3$,

i.e. **when $\boldsymbol{\dfrac{3-\sqrt{6}}{3}\leqslant a<1}$,**

the minimum value is $\boldsymbol{f(a+2)=a^3-3a+2}$.

(iii) When $a\leqslant 3\leqslant a+2$,

i.e. **when $\boldsymbol{1\leqslant a\leqslant 3}$,**

the minimum value is $\boldsymbol{f(3)=0}$.

(iv) **When $\boldsymbol{a>3}$,**

the minimum value is $\boldsymbol{f(a)=a^3-6a^2+9a}$.

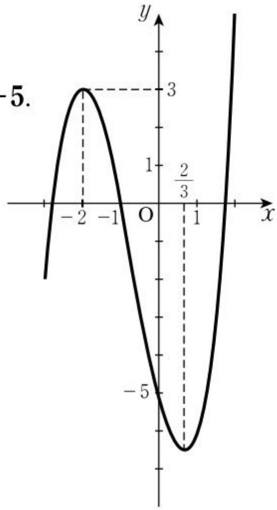
L 96b

2. Find the maximum value of $f(x)=x^3+2x^2-4x-5$ in the domain $t-2 \leqslant x \leqslant t$.

[Sol] $f'(x)=3x^2+4x-4=(3x-2)(x+2)$

$f(x)$ has a relative maximum at $x=-2$,

and a relative minimum at $x=\dfrac{2}{3}$.



(i) **When $t<-2$,**

the maximum value is $\boldsymbol{f(t)=t^3+2t^2-4t-5}$.

(ii) When $t-2 \leqslant -2 \leqslant t$,

i.e. **when $-2 \leqslant t \leqslant 0$,**

the maximum value is $\boldsymbol{f(-2)=3}$.

(iii) If $t>0$, first consider when both ends of the domain $t-2 \leqslant x \leqslant t$ give the same value:

$$f(t-2)=(t-2)^3+2(t-2)^2-4(t-2)-5$$
$$=t^3-4t^2+3$$

If $f(t-2)=f(t)$,

$$t^3-4t^2+3=t^3+2t^2-4t-5$$
$$2(3t^2-2t-4)=0$$

Since $t>0$, $t=\dfrac{1+\sqrt{13}}{3}$

Therefore, **when $0<t<\dfrac{1+\sqrt{13}}{3}$,**

the maximum value is $\boldsymbol{f(t-2)=t^3-4t^2+3}$.

(iv) **When $t \geqslant \dfrac{1+\sqrt{13}}{3}$,**

the maximum value is $\boldsymbol{f(t)=t^3+2t^2-4t-5}$.

L 97a KUMON

Maxima and Minima II

Ex.

Given that the minimum value of $y=x^2-6x+10$, $(a \leqslant x \leqslant a+2)$, is b, and b is a function of a, find the equation of this function, and then sketch its graph.

[Sol] Let $y=f(x)$

$f(x)=x^2-6x+10=(x-3)^2+1$

(i) When $a+2<\boxed{3}$, i.e. when $a<1$,

$b=f(a+2)=(a-1)^2+1 \quad \cdots①$

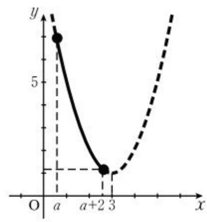
(ii) When $a \leqslant \boxed{3} \leqslant a+2$, i.e. when $1 \leqslant a \leqslant 3$,

$b=f(3)=1 \quad \cdots②$

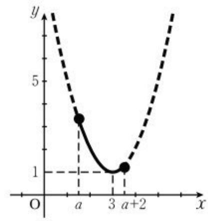
(iii) When $a>\boxed{3}$,

$b=f(a)=(a-3)^2+1 \quad \cdots③$

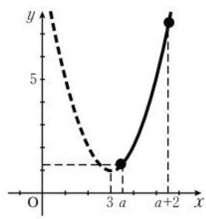
(i)



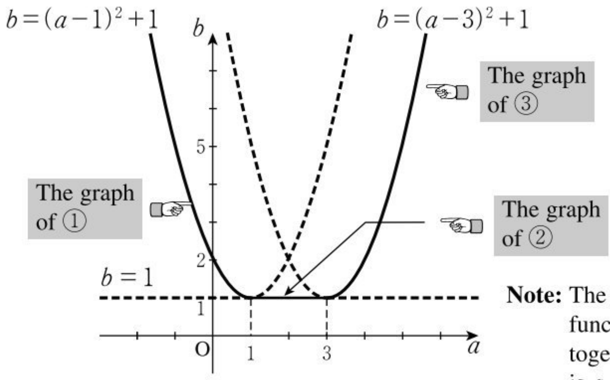
(ii)



(iii)



Therefore, sketching the graph:



Note: The graph on the left shows functions ①, ② and ③ all together. The horizontal axis is a, and the vertical axis is b.

Answers: All the answers are the same, 3.

L 97b

1. Given that the minimum value of $y=x^2-4x+3$, $(a \leqslant x \leqslant a+3)$, is b, and b is a function of a, find the equation of this function, and then sketch its graph.

[Sol] Let $y=f(x)$

$f(x)=x^2-4x+3=(x-2)^2-1$

(i) When $a+3<2$, i.e. **when $\boldsymbol{a<-1}$**,

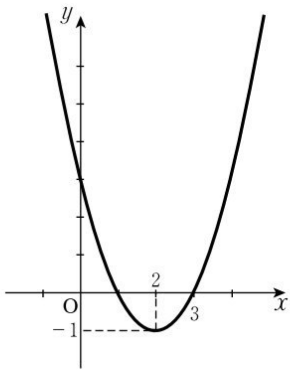
$\boldsymbol{b}=f(a+3)=\boldsymbol{(a+1)^2-1}$

(ii) When $a \leqslant 2 \leqslant a+3$, i.e. **when $\boldsymbol{-1 \leqslant a \leqslant 2}$**,

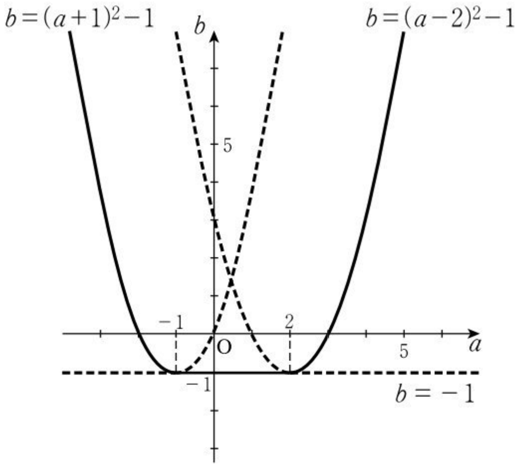
$\boldsymbol{b}=f(2)=\boldsymbol{-1}$

(iii) **When $\boldsymbol{a>2}$**,

$\boldsymbol{b}=f(a)=\boldsymbol{(a-2)^2-1}$



Therefore, sketching the graph:



L 98a KUMON

Maxima and Minima II

Ex.

Given that the maximum value of $y = x^3 - 6x^2 + 9x - 2$ in the domain $0 \leqslant x \leqslant a$, (where $a > 0$) is b, and b is a function of a, find the equation of this function, and then sketch its graph.

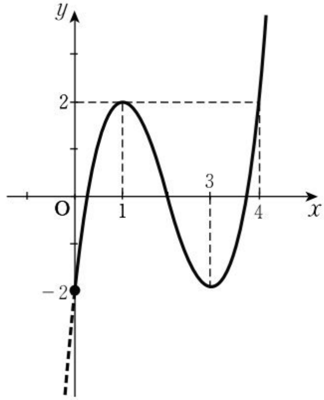
[Sol]

Let $y = f(x)$

$$f(x) = x^3 - 6x^2 + 9x - 2$$

$$f'(x) = 3x^2 - 12x + 9$$

$$= 3(x-3)(x-1)$$



Therefore, $f(x)$ has:

a relative maximum value of 2, at $x = 1$, and

a relative minimum value of -2, at $x = 3$.

Find the value of $x > 1$ where $f(x) = 2$.

$$x^3 - 6x^2 + 9x - 2 = 2$$

$$(x-1)^2(x-4) = 0$$

$$x = 4$$

(i) When $0 < a < \boxed{1}$,

$b = f(a) = a^3 - 6a^2 + 9a - 2 \quad \cdots ①$

(ii) When $\boxed{1} \leqslant a \leqslant \boxed{4}$,

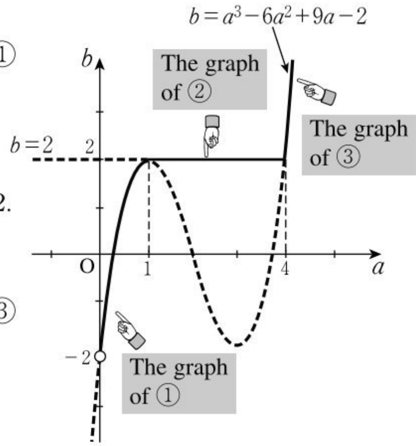
$b = f(1) = 2 \quad \cdots ②$

Note: When $a = 4$, then $b = f(1) = f(4) = 2$.

(iii) When $a > \boxed{4}$,

$b = f(a) = a^3 - 6a^2 + 9a - 2 \quad \cdots ③$

Therefore, sketching the graph:



Answers: in order 1, 1, 4, 4

L98b

1. Given that the maximum value of $y=2x^3-12x^2+18x-3$ in the domain $0 \leqslant x \leqslant a$, (where $a>0$), is b, and b is a function of a, find the equation of this function, and then sketch its graph.

[Sol] Let $y=f(x)$

$$f(x)=2x^3-12x^2+18x-3$$
$$f'(x)=6x^2-24x+18$$
$$=6(x-3)(x-1)$$

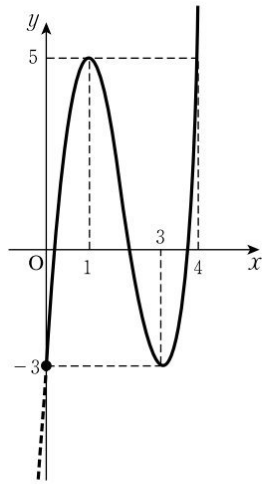
Therefore, $f(x)$ has:

a relative maximum value of 5, at $x=1$, and

a relative minimum value of -3, at $x=3$.

Find the value of $x>1$ where $f(x)=5$.

$$2x^3-12x^2+18x-3=5$$
$$x^3-6x^2+9x-4=0$$
$$(x-1)^2(x-4)=0$$
$$x=4$$



(i) **When $0<a<1$,**

$\boldsymbol{b=f(a)=2a^3-12a^2+18a-3}$

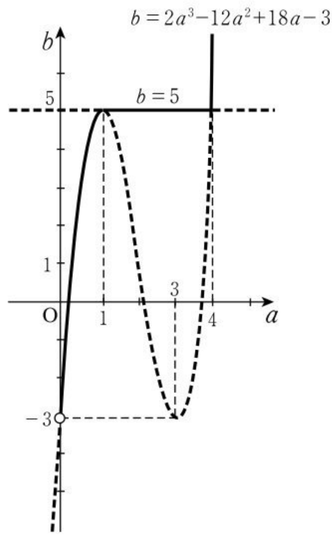
(ii) **When $1 \leqslant a \leqslant 4$,**

$\boldsymbol{b=f(1)=5}$

(iii) **When $a>4$,**

$\boldsymbol{b=f(a)=2a^3-12a^2+18a-3}$

Therefore, sketching the graph:



L 99a KUMON

Maxima and Minima II

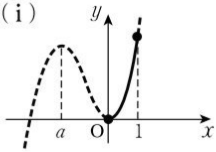
Ex.

Given that the minimum value of $y = 2x^3 - 3ax^2$ in the domain $0 \leqslant x \leqslant 1$, is b, and b is a function of a, find the equation of this function, and then sketch its graph.

[Sol] Let $y = f(x)$

$$f(x) = 2x^3 - 3ax^2$$
$$f'(x) = 6x^2 - 6ax = 6x(x - a)$$

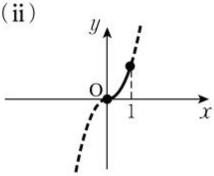
(i)



(i) When $a < \boxed{0}$,

from the graph, $b = f(0) = 0$.

(ii)

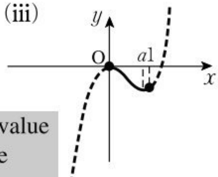


(ii) When $a = \boxed{0}$,

$f(x) = 2x^3$

from the graph, $b = f(0) = 0$.

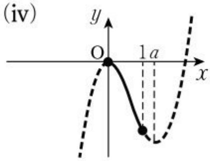
(iii)



(iii) When $\boxed{0} < a \leqslant \boxed{1}$, ☜ The minimum value is at the relative minimum value.

from the graph, $b = f(a) = -a^3$

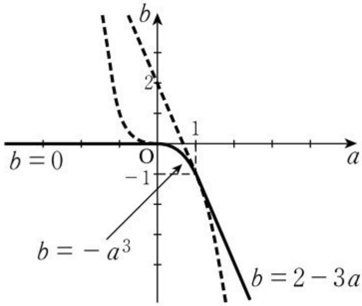
(iv)



(iv) When $a > \boxed{1}$, ☜ The minimum value is at the right end of the domain.

from the graph, $b = f(1) = 2 - 3a$.

Therefore, sketching the graph:



Answers: in order 0, 0, 0, 1, 1

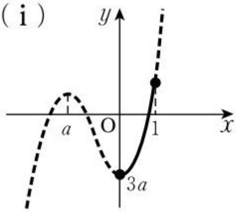
L99b

1. Given that the minimum value of $y = 2x^3 - 3ax^2 + 3a$ in the domain $0 \leqslant x \leqslant 1$, is b, and b is a function of a, find the equation of this function, and then sketch its graph.

[Sol] Let $y = f(x)$

$$f(x) = 2x^3 - 3ax^2 + 3a$$
$$f'(x) = 6x^2 - 6ax = 6x(x - a)$$

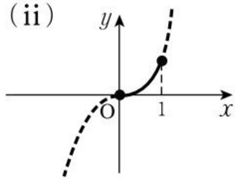
(i)



(i) **When $\boldsymbol{a < 0}$,**

from the graph, $\boldsymbol{b} = f(0) = \mathbf{3}\boldsymbol{a}$

(ii)

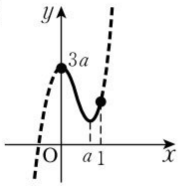


(ii) **When $\boldsymbol{a = 0}$,**

$$f(x) = 2x^3$$

from the graph, $\boldsymbol{b} = f(0) = \mathbf{0}$

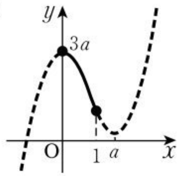
(iii)



(iii) **When $\boldsymbol{0 < a \leqslant 1}$,**

from the graph, $\boldsymbol{b} = f(a) = \boldsymbol{-a^3 + 3a}$

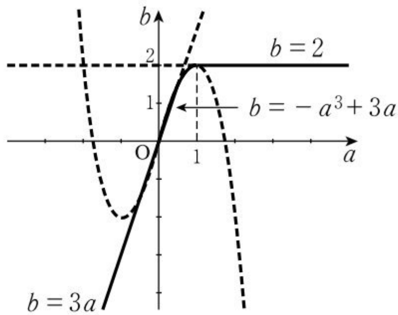
(iv)



(iv) **When $\boldsymbol{a > 1}$,**

from the graph, $\boldsymbol{b} = f(1) = \mathbf{2}$

Therefore, sketching the graph:



L 100a KUMON

Maxima and Minima II

1. Find the maximum value of $f(x)=x^3-3x+2$ in the domain $-a\leqslant x\leqslant a$ (where $a>0$).

[Sol] $f'(x)=3x^2-3$

$\qquad =3(x+1)(x-1)$

There is a relative maximum value of 4, at $x=-1$, and a relative minimum value of 0, at $x=1$.

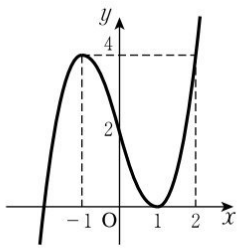
Find the value of $x>-1$ where $f(x)=4$.

$x^3-3x+2=4$

$x^3-3x-2=0$

$(x+1)^2(x-2)=0$

$x=2$



(i) **When $0<a<1$,**

the maximum value is $\boldsymbol{f(-a)=-a^3+3a+2}$.

(ii) **When $1\leqslant a\leqslant 2$,**

the maximum value is $\boldsymbol{f(-1)=4}$.

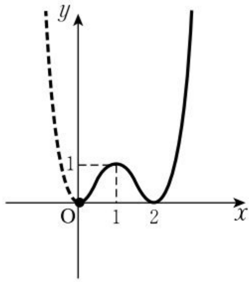
(iii) **When $a>2$,**

the maximum value is $\boldsymbol{f(a)=a^3-3a+2}$.

L 100b

2. Given that the maximum value of $y=x^2(x-2)^2$ in the domain $0 \leqslant x \leqslant a$ (where $a>0$), is b, and b is a function of a, find the equation of this function, and then, sketch its graph.

[Sol] Let $y=f(x)$



$$f(x)=x^2(x-2)^2=x^4-4x^3+4x^2$$
$$f'(x)=4x^3-12x^2+8x$$
$$=4x(x-2)(x-1)$$

Therefore, $f(x)$ has:

a relative maximum value of 1, at $x=1$, and

a relative minimum value of 0, at $x=0,\ 2$.

Find the value of $x>1$ where $f(x)=1$.

$$x^2(x-2)^2=1$$
$$x^4-4x^3+4x^2-1=0$$
$$(x-1)^2(x^2-2x-1)=0$$
$$x=1,\ 1\pm\sqrt{2}$$

Since $x>1$, $x=1+\sqrt{2}$

(i) **When $\boldsymbol{0<a<1}$,**

$\boldsymbol{b}=f(a)=\boldsymbol{a^2(a-2)^2}$

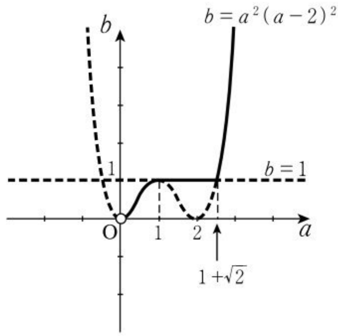
(ii) **When $\boldsymbol{1\leqslant a\leqslant 1+\sqrt{2}}$,**

$\boldsymbol{b}=f(1)=\boldsymbol{1}$

(iii) **When $\boldsymbol{a>1+\sqrt{2}}$,**

$\boldsymbol{b}=f(a)=\boldsymbol{a^2(a-2)^2}$

Therefore, sketching the graph:



L 101 a KUMON

Applications to Equations and Inequalities

> The real roots of an equation $f(x)=0$ correspond to points where the function $y=f(x)$ crosses the x-axis.

1. Find the number of real roots of the following equations by sketching.

Ex.

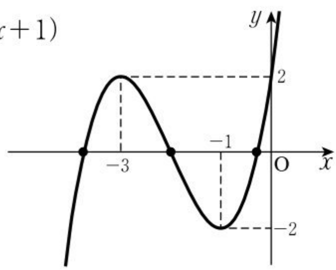
$x^3+6x^2+9x+2=0$

[Sol] Let $f(x)=x^3+6x^2+9x+2$

$f'(x)=3x^2+12x+9=3(x+3)(x+1)$

x	$\cdots$	-3	$\cdots$	-1	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	2	$\searrow$	-2	$\nearrow$

$f(0)=2$



From the graph, we can find that the equation has 3 real roots.

The curve intersects the x-axis at 3 different points.

(1) $x^3+6x^2+9x-2=0$

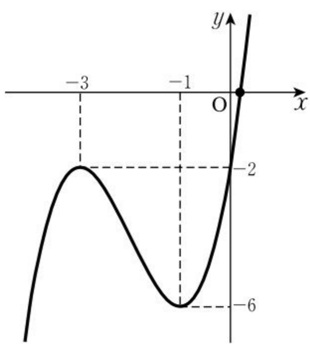
[Sol] Let $f(x)=x^3+6x^2+9x-2$

$f'(x)=3x^2+12x+9$

$=3(x+3)(x+1)$

x	$\cdots$	-3	$\cdots$	-1	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	-2	$\searrow$	-6	$\nearrow$

$f(0)=-2$



From the graph,
the equation has **1 real root**.

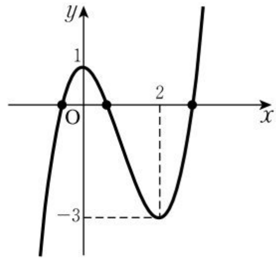
L101b

(2)　$x^3-3x^2+1=0$

[Sol] Let $f(x)=x^3-3x^2+1$

$$f'(x)=3x^2-6x$$
$$=3x(x-2)$$

x	$\cdots$	0	$\cdots$	2	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	1	$\searrow$	-3	$\nearrow$



From the graph,

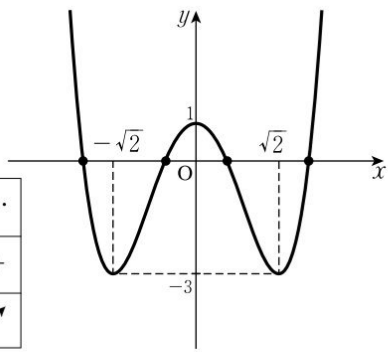
the equation has **3 real roots**.

(3)　$x^4-4x^2+1=0$

[Sol] Let $f(x)=x^4-4x^2+1$

$$f'(x)=4x^3-8x$$
$$=4x(x^2-2)$$
$$=4x(x+\sqrt{2})(x-\sqrt{2})$$

x	$\cdots$	$-\sqrt{2}$	$\cdots$	0	$\cdots$	$\sqrt{2}$	$\cdots$
$f'(x)$	$-$	0	$+$	0	$-$	0	$+$
$f(x)$	$\searrow$	-3	$\nearrow$	1	$\searrow$	-3	$\nearrow$



From the graph,

the equation has **4 real roots**.

L 102a KUMON

Applications to Equations and Inequalities

1. Find the number of positive roots and the number of negative roots by sketching each of the following equations.

Ex.

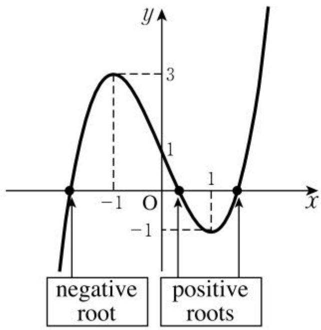
$x^3-3x+1=0$

[Sol] Let $f(x)=x^3-3x+1$

$f'(x)=3x^2-3=3(x+1)(x-1)$

x	$\cdots$	-1	$\cdots$	1	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	3	$\searrow$	-1	$\nearrow$

$f(0)=1$



From the graph,

there are 2 positive roots and 1 negative root.

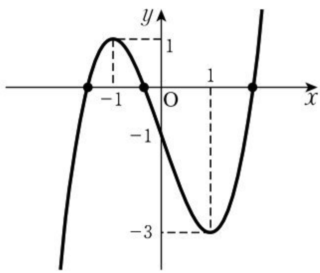
(1) $x^3-3x-1=0$

[Sol] Let $f(x)=x^3-3x-1$

$f'(x)=3x^2-3=3(x+1)(x-1)$

x	$\cdots$	-1	$\cdots$	1	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	1	$\searrow$	-3	$\nearrow$

$f(0)=-1$



From the graph,

there is **1 positive root and 2 negative roots**.

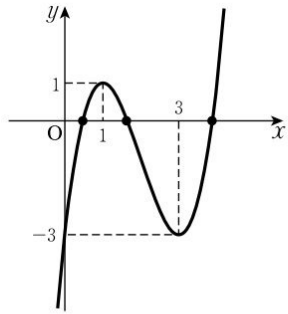
L 102b

(2) $x^3-6x^2+9x-3=0$

[Sol] Let $f(x)=x^3-6x^2+9x-3$

$f'(x)=3x^2-12x+9=3(x-1)(x-3)$

x	$\cdots$	1	$\cdots$	3	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	1	$\searrow$	-3	$\nearrow$



$f(0)=-3$

From the graph,

there are **3 positive roots**.

(3) $x^4-2x^2-1=0$

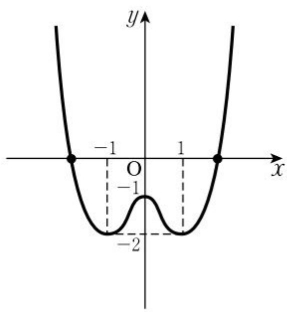
[Sol] Let $f(x)=x^4-2x^2-1$

$f'(x)=4x^3-4x$

$=4x(x^2-1)$

$=4x(x+1)(x-1)$

x	$\cdots$	-1	$\cdots$	0	$\cdots$	1	$\cdots$
$f'(x)$	$-$	0	$+$	0	$-$	0	$+$
$f(x)$	$\searrow$	-2	$\nearrow$	-1	$\searrow$	-2	$\nearrow$



$f(0)=-1$

From the graph,

there is **1 positive root and 1 negative root**.

L 103a KUMON

Applications to Equations and Inequalities

Ex.

Find the number of real roots of the cubic equation $x^3-3x+1-a=0$ by sketching the graph. (Consider different values of a.)

[Sol]

From $x^3-3x+1=a$, ☜ Rearrange the original equation.

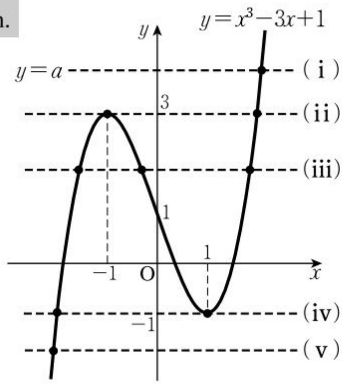
let

$$\begin{cases} y=a & \cdots ① \\ y=x^3-3x+1 & \cdots ② \end{cases}$$

From ②,

$$y'=3x^2-3=3(x+1)(x-1)$$

x	$\cdots$	-1	$\cdots$	1	$\cdots$
y'	$+$	0	$-$	0	$+$
y	↗	3	↘	-1	↗



From the graph, we can find the number of common points of ① and ②.

The number of real roots:

When $a>3$, there is/are $\boxed{\mathbf{1}}$ real root(s). ☜ From the graph of (i)

When $a=3$, there is/are $\boxed{\mathbf{2}}$ real root(s). ☜ From the graph of (ii)

When $\boxed{\mathbf{-1}}<a<3$, there is/are $\boxed{\mathbf{3}}$ real root(s). ☜ From the graph of (iii)

When $a=\boxed{\mathbf{-1}}$, there is/are $\boxed{\mathbf{2}}$ real root(s). ☜ From the graph of (iv)

When $a<\boxed{\mathbf{-1}}$, there is/are $\boxed{\mathbf{1}}$ real root(s). ☜ From the graph of (v)

Answers: in order 1, 2, −1, 3, −1, 2, −1, 1

L 103b

1. Find the number of real roots of the quartic equation $2x^4-3x^2-a=0$ by sketching the graph. (Consider different values of a.)

[Sol] From $2x^4-3x^2=a$,

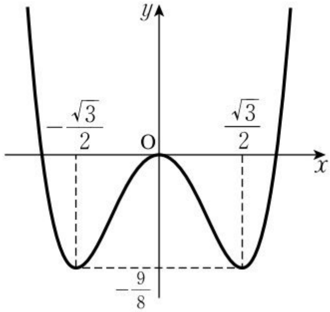
let

$$\begin{cases} y=a & \cdots ① \\ y=2x^4-3x^2 & \cdots ② \end{cases}$$

From ②,

$$y'=8x^3-6x=2x(4x^2-3)$$
$$=2x(2x+\sqrt{3})(2x-\sqrt{3})$$

x	$\cdots$	$-\frac{\sqrt{3}}{2}$	$\cdots$	0	$\cdots$	$\frac{\sqrt{3}}{2}$	$\cdots$
y'	$-$	0	$+$	0	$-$	0	$+$
y	↘	$-\frac{9}{8}$	↗	0	↘	$-\frac{9}{8}$	↗



From the graph, we can find the number of common points of ① and ②.

The number of real roots:

$$\begin{cases} \textbf{When } a>0\textbf{, there are 2 real roots.} \\ \textbf{When } a=0\textbf{, there are 3 real roots.} \\ \textbf{When } -\frac{9}{8}<a<0\textbf{, there are 4 real roots.} \\ \textbf{When } a=-\frac{9}{8}\textbf{, there are 2 real roots.} \\ \textbf{When } a<-\frac{9}{8}\textbf{, there are 0 real roots.} \end{cases}$$

L 104a KUMON

Applications to Equations and Inequalities

Ex.

Given the cubic equation $x^3-3x-a=0$, find the values of a for which the equation has the following solutions:

(1) 2 different real roots (one repeated root and one simple root)

(2) 1 positive real root only

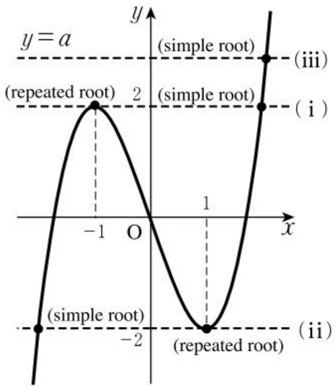
[Sol] From $x^3-3x=a$,

let

$$\begin{cases} y=a & \cdots① \\ y=x^3-3x & \cdots② \end{cases}$$

From ②, $y'=3(x+1)(x-1)$

x	$\cdots$	-1	$\cdots$	1	$\cdots$
y'	$+$	0	$-$	0	$+$
y	$\nearrow$	2	$\searrow$	-2	$\nearrow$



(1) From the graph,

$a=\pm\boxed{2}$ When ① and ② touch. (shown on the graph as (i) and (ii))

(2) From the graph,

$a>\boxed{2}$ When ① and ② intersect at one point, with $x>0$. (shown on the graph as (iii))

Answers: All the answers are the same, 2.

Note: When $a<-2$, the equation has 1 real root only, but it is a negative root.

Definitions:

Repeated root — Given a polynomial equation $f(x)=0$ with a factor $(x-a)^n$, where n is an integer >1, we say that $x=a$ is a repeated root.

Simple root — A simple root is a root that is not a repeated root.

L 104b

1. Given the quartic equation $x^4 - 2x^2 - a = 0$, find the values of a for which the equation has the following solutions:

(1) 3 different real roots (one repeated root and two simple roots)

(2) 4 real roots

[Sol] From $x^4 - 2x^2 = a$,

let

$$\begin{cases} y = a & \cdots① \\ y = x^4 - 2x^2 & \cdots② \end{cases}$$

From ②, $y' = 4x^3 - 4x = 4x(x+1)(x-1)$

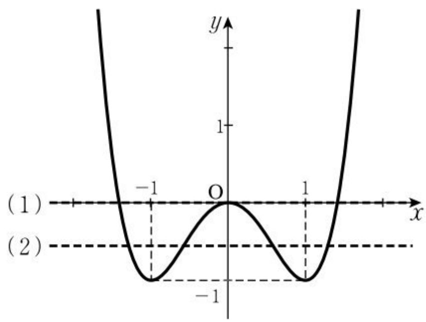
x	$\cdots$	-1	$\cdots$	0	$\cdots$	1	$\cdots$
y'	$-$	0	$+$	0	$-$	0	$+$
y	$\searrow$	-1	$\nearrow$	0	$\searrow$	-1	$\nearrow$

(1) From the graph,

$\boldsymbol{a = 0}$

(2) From the graph,

$\boldsymbol{-1 < a < 0}$



L 105a KUMON

Applications to Equations and Inequalities

Ex.

Find the range of values of the constant k for which the cubic equation $x^3-6x^2+9x+k=0$ has the following solutions:

(1) 2 different real roots (one repeated root and one simple root)

(2) 3 real roots

[Sol]

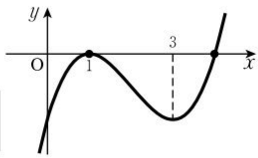
Let $f(x)=x^3-6x^2+9x+k$

$f'(x)=3x^2-12x+9=3(x-1)(x-3)$

x	$\cdots$	1	$\cdots$	3	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	$k+4$	$\searrow$	k	$\nearrow$

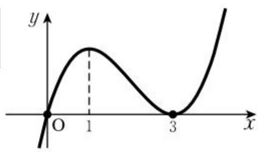
(1) From the graph,

$f(1)=k+4=0 \quad \cdots①$ When the relative maximum value $=0$.



or

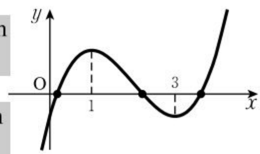
$f(3)=k=0 \quad \cdots②$ When the relative minimum value $=0$.



From ① and ②, $k=\boxed{-4}$, $\boxed{0}$

(2) From the graph,

$f(1)=k+4>0 \quad \cdots①$ The relative maximum value must be >0



$f(3)=k<0 \quad \cdots②$ The relative minimum value must be <0

From ① and ②, $\boxed{-4}<k<\boxed{0}$

Answers: -4, 0 (in any order) -4, 0 (in order)

L 105b

1. Using the method shown on side a, find the range of values of the constant a for which the cubic equation $x^3-3x-a=0$ has the following solutions:

(1) 2 different real roots (one repeated root and one simple root)

(2) 3 real roots

[Sol] Let $f(x) = x^3-3x-a$

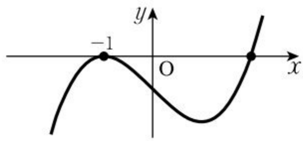
$f'(x)=3x^2-3=3(x+1)(x-1)$

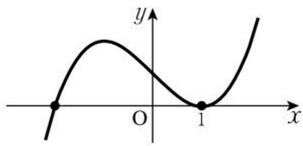
x	$\cdots$	-1	$\cdots$	1	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	$2-a$	$\searrow$	$-2-a$	$\nearrow$

(1) From the graph,

$2-a=0$ or $-2-a=0$

Therefore, $\boldsymbol{a=\pm 2}$

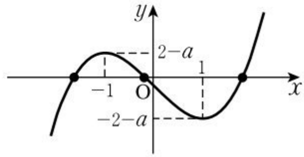




(2) From the graph,

$-2-a<0<2-a$

Therefore, $\boldsymbol{-2<a<2}$



We can also find the roots by obtaining the points of intersection between $y=x^3-3x$ and $y=a$. (Compare the examples on L104a and L105b.)

L 106a KUMON

Applications to Equations and Inequalities

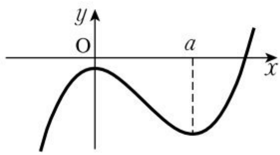
1. For the equation $2x^3-3ax^2+a^3-2a=0$ (where $a\neq 0$), find the range of values of a for which the equation has 1 real root only.

[Sol] Let $f(x)=2x^3-3ax^2+a^3-2a$

$f'(x)=6x^2-6ax=6x(x-a)$

(i) When $a>0$,

x	$\cdots$	0	$\cdots$	a	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	a^3-2a	$\searrow$	$-2a$	$\nearrow$



The relative minimum $(-2a)$ is negative, so for there to be only one root, the relative maximum (a^3-2a) must be negative too:

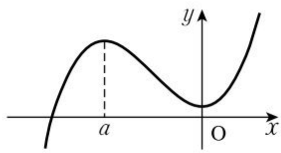
$a^3-2a=a(a+\sqrt{2})(a-\sqrt{2})<0$

$a<-\sqrt{2},\ 0<a<\sqrt{2}$

Therefore, since $a>0$, $0<a<\sqrt{2}$

(ii) When $a<0$,

x	$\cdots$	a	$\cdots$	0	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	$-2a$	$\searrow$	a^3-2a	$\nearrow$



The relative maximum $(-2a)$ is positive, so the relative minimum (a^3-2a) must be positive too:

$a(a+\sqrt{2})(a-\sqrt{2})>0$

$-\sqrt{2}<a<0,\ a>\sqrt{2}$

Therefore, since $a<0$, $-\sqrt{2}<a<0$

From (i) and (ii), $\boldsymbol{-\sqrt{2}<a<0,\ 0<a<\sqrt{2}}$

L 106b

2. Find the number of points of intersection between $y = x^3 - \frac{3}{2}x^2 - 5x + 1$ and $y = x - a$ by considering different values of a.

[Sol] From $x^3 - \frac{3}{2}x^2 - 5x + 1 = x - a$,

let $f(x) = x^3 - \frac{3}{2}x^2 - 6x + 1 + a$

$f'(x) = 3x^2 - 3x - 6 = 3(x+1)(x-2)$

x	$\cdots$	-1	$\cdots$	2	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	$\frac{9}{2}+a$	$\searrow$	$a-9$	$\nearrow$

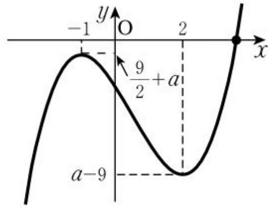
Relative maximum: $f(-1) = \frac{9}{2} + a$

Relative minimum: $f(2) = a - 9$

Therefore, determining the number of points of intersection:

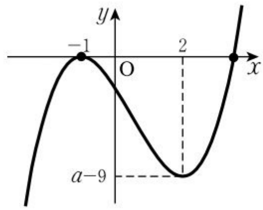
(i) When $\frac{9}{2} + a < 0$, i.e. **when $\boldsymbol{a < -\frac{9}{2}}$**, **there is 1 point of intersection**.

(i)



(ii) When $\frac{9}{2} + a = 0$, i.e. **when $\boldsymbol{a = -\frac{9}{2}}$**, **there are 2 points of intersection**.

(ii)

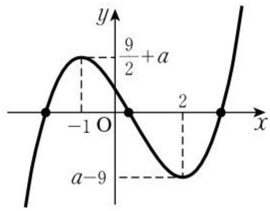


(iii) When $a - 9 < 0 < \frac{9}{2} + a$,

i.e. **when $\boldsymbol{-\frac{9}{2} < a < 9}$**,

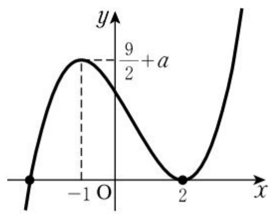
there are 3 points of intersection.

(iii)



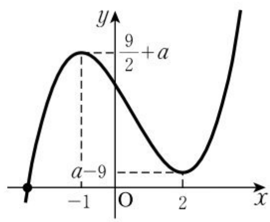
(iv) When $a - 9 = 0$, i.e. **when $\boldsymbol{a = 9}$**, **there are 2 points of intersection**.

(iv)



(v) When $a - 9 > 0$, i.e. **when $\boldsymbol{a > 9}$**, **there is 1 point of intersection**.

(v)



L 107a KUMON

Applications to Equations and Inequalities

Creating a Variation Table (i.e. a table showing how a function increases and decreases) can help when proving inequalities.

Ex.

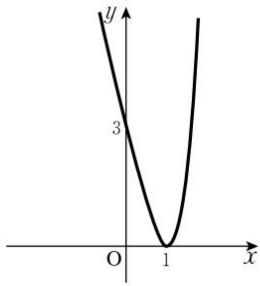
Prove that $x^4+3 \geqq 4x$.

[Sol] Let $f(x)=x^4+3-4x$

$f'(x)=4x^3-4=4(x-1)(x^2+x+1)$

Since $x^2+x+1>0$ is always true,

x	$\cdots$	1	$\cdots$
$f'(x)$	$-$	0	$+$
$f(x)$	$\searrow$	0	$\nearrow$



The minimum value of $f(x)$ is 0. ☜ $f(x)$ cannot be a negative value.

Therefore, $f(x) \geqq 0$

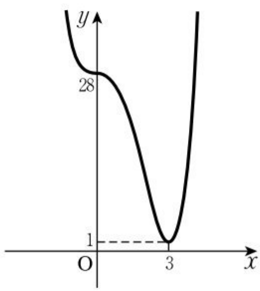
Thus, $x^4+3 \geqq 4x$

1. Prove that $x^4+28>4x^3$.

[Sol] Let $f(x)=x^4+28-4x^3$

$f'(x)=4x^3-12x^2=4x^2(x-3)$

x	$\cdots$	0	$\cdots$	3	$\cdots$
$f'(x)$	$-$	0	$-$	0	$+$
$f(x)$	$\searrow$	28	$\searrow$	1	$\nearrow$



The minimum value of $f(x)$ is 1.

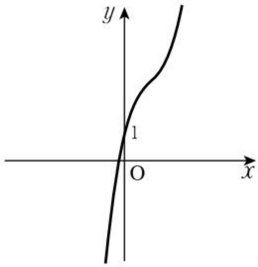
Therefore, $f(x)>0$

Thus, $x^4+28>4x^3$

L 107b

Ex.

Prove that when $x>0$, $x^3-3x^2+4x+1>0$.



[Sol] Let $f(x) = x^3-3x^2+4x+1$

$$f'(x) = 3x^2-6x+4$$
$$= 3(x^2-2x+1)+1$$
$$= 3(x-1)^2+1>0$$

Therefore, $f(x)$ is monotonically increasing.

As $f(0)=1$,

$f(x)>1$ for $x>0$.

Thus,

when $x>0$, $x^3-3x^2+4x+1>0$

2. Prove that when $x>0$, $x^3+7x+1>3x^2$.

[Sol] Let $f(x) = x^3+7x+1-3x^2$

$$f'(x) = 3x^2-6x+7$$
$$= 3(x-1)^2+4>0$$

Therefore, $f(x)$ is monotonically increasing.

As $f(0)=1$,

$f(x)>1$ for $x>0$.

Thus,

when $x>0$, $x^3+7x+1>3x^2$

L 108a KUMON

Applications to Equations and Inequalities

1. Prove that when $x > 0$, $\left(x+\frac{1}{2}\right)^3 > 3x^2+\frac{1}{8}$.

[Sol] Let $f(x) = \left(x+\frac{1}{2}\right)^3 - 3x^2 - \frac{1}{8}$

$$f(x) = x^3 - \frac{3}{2}x^2 + \frac{3}{4}x$$

$$f'(x) = 3x^2 - 3x + \frac{3}{4}$$

$$= 3\left(x-\frac{1}{2}\right)^2 \geqslant 0$$

Therefore, $f(x)$ is monotonically increasing.

As $f(0) = 0$,

$f(x) > 0$ for $x > 0$.

Thus,

when $x > 0$, $\left(x+\frac{1}{2}\right)^3 > 3x^2+\frac{1}{8}$

L 108b

2. Prove that when $x>0$, $2x^3-9x^2+12x+1>0$.

[Sol] Let $f(x)=2x^3-9x^2+12x+1$

$$f'(x)=6x^2-18x+12$$
$$=6(x-2)(x-1)$$

x	0	$\cdots$	1	$\cdots$	2	$\cdots$
$f'(x)$	$+$	$+$	0	$-$	0	$+$
$f(x)$	1	$\nearrow$	6	$\searrow$	5	$\nearrow$

As $f(0)=1$ and $f(2)=5$,

$f(x)>0$ for $x>0$.

Thus,

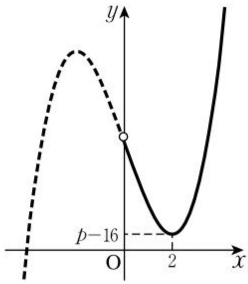
when $x>0$, $2x^3-9x^2+12x+1>0$

L 109a KUMON

Applications to Equations and Inequalities

Ex.

Find the conditions on the real number p for which $x^3-12x+p>0$, where $x>0$.



[Sol] Let $f(x)=x^3-12x+p$

$f'(x)=3x^2-12=3(x+2)(x-2)$

x	0	$\cdots$	2	$\cdots$
$f'(x)$		$-$	0	$+$
$f(x)$		$\searrow$	$p-16$	$\nearrow$

Therefore, from $p-16>0$,

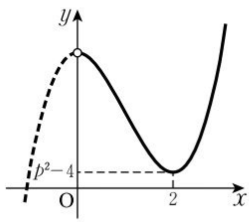
$p>16$

In order for $f(x)>0$ to be true, the relative minimum value must be greater than zero.

1. Find the conditions on the real number p for which $x^3-3x^2+p^2>0$, where $x>0$.

[Sol] Let $f(x)=x^3-3x^2+p^2$

$f'(x)=3x^2-6x=3x(x-2)$



x	0	$\cdots$	2	$\cdots$
$f'(x)$		$-$	0	$+$
$f(x)$		$\searrow$	p^2-4	$\nearrow$

Therefore, from $p^2-4>0$,

$\boldsymbol{p<-2,\ p>2}$

L 109b

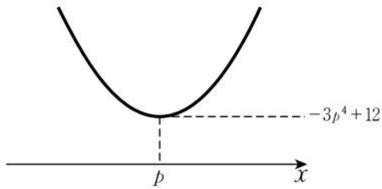
2. Find the conditions on the real number p for which $x^4-4p^3x+12>0$ for all real numbers x.

[Sol] Let $f(x)=x^4-4p^3x+12$

$$f'(x)=4x^3-4p^3=4(x-p)(x^2+px+p^2)$$

$$x^2+px+p^2=\left(x+\frac{1}{2}p\right)^2+\frac{3}{4}p^2\geqslant 0 \text{ for all real numbers } x.$$

x	$\cdots$	p	$\cdots$
$f'(x)$	$-$	0	$+$
$f(x)$	$\searrow$	$-3(p^4-4)$	$\nearrow$



$$f(p)=p^4-4p^4+12=-3(p^4-4)$$

From $-3(p^4-4)>0$,

$$(p^2+2)(p+\sqrt{2})(p-\sqrt{2})<0$$

Therefore,

$$\boldsymbol{-\sqrt{2}<p<\sqrt{2}}$$

L110a KUMON

Applications to Equations and Inequalities

1. Find the range of values of the real number m for which the equation $x^3-3x^2-9x-m=0$ has 2 different positive roots and 1 negative root.

[Sol] From $x^3-3x^2-9x=m$,

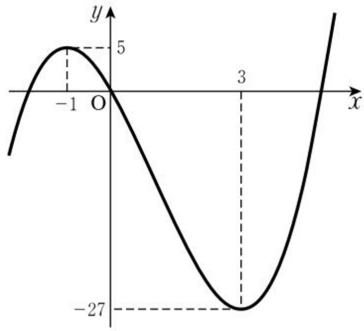
let

$$\begin{cases} y=m & \cdots ① \\ y=x^3-3x^2-9x & \cdots ② \end{cases}$$

Therefore, from ②,

$$y'=3x^2-6x-9=3(x-3)(x+1)$$

x	$\cdots$	-1	$\cdots$	3	$\cdots$
y'	$+$	0	$-$	0	$+$
y	↗	5	↘	-27	↗



From the graph,

$$\mathbf{-27<m<0}$$

L 110b

2. Prove that $\frac{1}{4}(x^4-1)+\frac{1}{2}(x^2-1)\leqslant\frac{2}{3}(x^3-1)$, when $0\leqslant x\leqslant 1$.

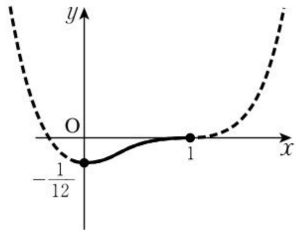
[Sol] Let $f(x)=\frac{1}{4}(x^4-1)+\frac{1}{2}(x^2-1)-\frac{2}{3}(x^3-1)$

$$f(x)=\frac{1}{4}x^4-\frac{2}{3}x^3+\frac{1}{2}x^2-\frac{1}{12}$$

$$f'(x)=x^3-2x^2+x$$

$$=x(x^2-2x+1)=x(x-1)^2$$

x	0	$\cdots$	1
$f'(x)$	0	$+$	0
$f(x)$	$-\frac{1}{12}$	$\nearrow$	0



From the graph,

when $0\leqslant x\leqslant 1$, $\frac{1}{4}(x^4-1)+\frac{1}{2}(x^2-1)\leqslant\frac{2}{3}(x^3-1)$

L IIIa KUMON

Indefinite and Definite Integrals

A function that, when differentiated, becomes $f(x)$ is called an ***indefinite integral*** of $f(x)$.

1. Complete the following statements regarding functions $F(x)$ and $f(x)$.

Ex.

$F(x)=\dfrac{1}{3}x^3$ is an indefinite integral (or antiderivative) of $f(x)=x^2$. ☜ From $\left(\dfrac{1}{3}x^3\right)'=x^2$

(1) $F(x)=\dfrac{1}{2}x^2$ is an indefinite integral of $f(x)=\boxed{x}$. ☜ $\left(\dfrac{1}{2}x^2\right)'=x$

(2) $F(x)=\dfrac{1}{4}x^4$ is an indefinite integral of $f(x)=\boxed{x^3}$.

(3) $F(x)=x^2$ is an indefinite integral of $f(x)=\boxed{2x}$.

2. Complete the following statements regarding functions $G(x)$ and $f(x)$.

(1) $G(x)=\dfrac{1}{3}x^3+1$ is an indefinite integral of $f(x)=\boxed{x^2}$. ☜ $\left(\dfrac{1}{3}x^3+1\right)'=x^2$

(2) $G(x)=\dfrac{1}{3}x^3-4$ is an indefinite integral of $f(x)=\boxed{x^2}$.

(3) $G(x)=\dfrac{1}{3}x^3+a$ is an indefinite integral of $f(x)=\boxed{x^2}$.

3. Fill in the blank box.

If $F(x)=\dfrac{1}{3}x^3$, and C is a constant, all the functions $G(x)$ in question 2 can be expressed as $G(x)=F(x)+C$.

In this case, all the functions $G(x)$ above are indefinite integrals of $f(x)=\boxed{x^2}$.

Generally, if C is a constant and $F(x)$ is an antiderivative of $f(x)$, then $F(x)+C$ is called the indefinite integral of $f(x)$.

The indefinite integral of $f(x)$ is written as $\int f(x)dx$ and is read as: "the integral of $f(x)$".
When one of the indefinite integrals of $f(x)$ is $F(x)$,
$\int f(x)dx = F(x) + C$ (C is called the ***constant of integration.***)

4. Evaluate the following indefinite integrals.

Ex.

$\int x^2 dx = \frac{1}{3}x^3 + C$ — For an indefinite integral, don't forget to write $+C$.

(1) $\int x^3 dx = \boldsymbol{\frac{1}{4}x^4 + C}$ — $\left(\frac{1}{4}x^4\right)' = x^3$

(2) $\int x^4 dx = \boldsymbol{\frac{1}{5}x^5 + C}$

(3) $\int x^5 dx = \boldsymbol{\frac{1}{6}x^6 + C}$

(4) $\int 2x dx = \boldsymbol{x^2 + C}$ — $(x^2)' = 2x$

(5) $\int 3x^2 dx = \boldsymbol{x^3 + C}$

(6) $\int 1 dx = \boldsymbol{x + C}$

➢ When n is a positive integer or zero, then $(x^{n+1})' = (n+1)x^n$

$\left(\frac{1}{n+1}x^{n+1}\right)' = x^n$ — From the example and questions (1), (2) and (3)

Therefore, we can deduce the following formula for the indefinite integral of x^n.

Formula

When n is a positive integer or zero,

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C \quad (C \text{ is the constant of integration.})$$

Notation:
- The process of finding integrals is called ***integration***.
- In question (6), $\int 1dx$ can also be written as $\int dx$.

L 112a KUMON

Indefinite and Definite Integrals

To calculate indefinite integrals, we can use the following properties:

Properties of Indefinite Integrals

1. $\int kf(x)dx = k\int f(x)dx$ (k is a constant.)

2. $\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$

3. $\int [f(x)-g(x)]dx = \int f(x)dx - \int g(x)dx$

1. Evaluate the following indefinite integrals.

Ex.

$\int 2x^2 dx = \frac{2}{3}x^3 + C$ ☞ (From property **1.** above) $= 2\int x^2 dx = 2\cdot\left(\frac{1}{3}x^3 + C\right) = \frac{2}{3}x^3 + C$

(The constant of integration can be written simply as C instead of $2C$.)

(1) $\int 3x^3 dx = \boldsymbol{\frac{3}{4}x^4 + C}$

(2) $\int (-2x^2)dx = \boldsymbol{-\frac{2}{3}x^3 + C}$

(3) $\int 6x^2 dx = \boldsymbol{2x^3 + C}$

(4) $\int ax^2 dx = \boldsymbol{\frac{a}{3}x^3 + C}$

(5) $\int 2t^2 dt = \boldsymbol{\frac{2}{3}t^3 + C}$

(6) $\int 8t^3 dt = \boldsymbol{2t^4 + C}$

(7) $\int (-3)dx = \boldsymbol{-3x + C}$

(8) $\int a dx = \boldsymbol{ax + C}$

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2. Evaluate the following indefinite integrals.

Ex.

$\int(3x^2-4x+3)dx = x^3-2x^2+3x+C$ ☜ (From properties **1**, **2** and **3** on side a) $= 3\int x^2dx - 4\int xdx + 3\int dx$

(Only a single constant of integration C is needed.)

(1) $\int(4x+3)dx = \boldsymbol{2x^2+3x+C}$

(2) $\int(x^2+3x+1)dx = \boldsymbol{\frac{1}{3}x^3+\frac{3}{2}x^2+x+C}$

(3) $\int(t^2-3t+1)dt = \boldsymbol{\frac{1}{3}t^3-\frac{3}{2}t^2+t+C}$

(4) $\int(x^2+2x+a)dx = \boldsymbol{\frac{1}{3}x^3+x^2+ax+C}$

(5) $\int(px^3+qx^2+rx)dx = \boldsymbol{\frac{p}{4}x^4+\frac{q}{3}x^3+\frac{r}{2}x^2+C}$

(6) $\int(2x+a^2)dx = \boldsymbol{x^2+a^2x+C}$

(7) $\int(s^2-t^2)dt = \boldsymbol{s^2t-\frac{1}{3}t^3+C}$

(8) $\int(s^2-t^2)ds = \boldsymbol{\frac{1}{3}s^3-t^2s+C}$

L 113a KUMON

Indefinite and Definite Integrals

Evaluate the following indefinite integrals.

(1) $\int (x+1)(x+2)dx = \int (x^2+3x+2)dx$

$$= \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x + C$$

(2) $\int (2x+1)^2 dx = \int (4x^2+4x+1)dx$

$$= \frac{4}{3}x^3 + 2x^2 + x + C$$

(3) $\int (x+1)^3 dx = \int (x^3+3x^2+3x+1)dx$

$$= \frac{1}{4}x^4 + x^3 + \frac{3}{2}x^2 + x + C$$

(4) $\int (x+1)(x^2-x+1)dx = \int (x^3+1)dx$

$$= \frac{1}{4}x^4 + x + C$$

(5) $\int (x+a)(x+b)dx = \int [x^2+(a+b)x+ab]dx$

$$= \frac{1}{3}x^3 + \frac{a+b}{2}x^2 + abx + C$$

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(6) $\int (x^2+2)^2 dx = \int (x^4+4x^2+4)dx$

$$= \boldsymbol{\frac{1}{5}x^5+\frac{4}{3}x^3+4x+C}$$

(7) $\int x(x-1)^2 dx = \int (x^3-2x^2+x)dx$

$$= \boldsymbol{\frac{1}{4}x^4-\frac{2}{3}x^3+\frac{1}{2}x^2+C}$$

(8) $\int (t+2)^2(t+1)dt = \int (t^2+4t+4)(t+1)dt$

$$= \int (t^3+5t^2+8t+4)dt$$

$$= \boldsymbol{\frac{1}{4}t^4+\frac{5}{3}t^3+4t^2+4t+C}$$

(9) $\int (x+1)^2 dx - \int (x-1)^2 dx = \int (x^2+2x+1)dx - \int (x^2-2x+1)dx$

$$= \frac{1}{3}x^3+x^2+x-\left(\frac{1}{3}x^3-x^2+x\right)+C$$

$$= \boldsymbol{2x^2+C}$$

Alternative Solution
Use property **3.** from L112a, in reverse

$$= \int [(x+1)^2-(x-1)^2]dx$$

$$= \int 4x dx = \boldsymbol{2x^2+C}$$

L 114a KUMON

Indefinite and Definite Integrals

1. Find the function $F(x)$ that satisfies the following conditions.

Ex.

$F'(x) = x^2 + 3x$, $F(1) = 2$

[Sol] $F(x) = \int (x^2 + 3x)dx = \frac{1}{3}x^3 + \frac{3}{2}x^2 + C$

From $F(1) = \frac{1}{3} + \frac{3}{2} + C = 2$, $C = \frac{1}{6}$

Therefore, $F(x) = \frac{1}{3}x^3 + \frac{3}{2}x^2 + \frac{1}{6}$

(1) $F'(x) = -x$, $F(0) = -1$

[Sol] $F(x) = \int (-x)dx = -\frac{1}{2}x^2 + C$

$F(0) = C = -1$

Therefore, $\boldsymbol{F(x) = -\frac{1}{2}x^2 - 1}$

(2) $F'(x) = x^3 - 3x$, $F(1) = 0$

[Sol] $F(x) = \int (x^3 - 3x)dx = \frac{1}{4}x^4 - \frac{3}{2}x^2 + C$

From $F(1) = \frac{1}{4} - \frac{3}{2} + C = 0$, $C = \frac{5}{4}$

Therefore, $\boldsymbol{F(x) = \frac{1}{4}x^4 - \frac{3}{2}x^2 + \frac{5}{4}}$

L114b

2. Find the function $F(x)$ that satisfies the following conditions when $F'(x)=f(x)$ is true.

(1) $f(x)=-2x+3$, $F(1)=0$

[Sol] $F(x)=\int(-2x+3)dx=-x^2+3x+C$

From $F(1)=-1+3+C=0$, $C=-2$

Therefore, $\boldsymbol{F(x)=-x^2+3x-2}$

(2) $f(x)=x^2+3x$, $F(1)=\frac{4}{3}$

[Sol] $F(x)=\int(x^2+3x)dx=\frac{1}{3}x^3+\frac{3}{2}x^2+C$

From $F(1)=\frac{1}{3}+\frac{3}{2}+C=\frac{4}{3}$, $C=-\frac{1}{2}$

Therefore, $\boldsymbol{F(x)=\frac{1}{3}x^3+\frac{3}{2}x^2-\frac{1}{2}}$

(3) $f(x)=3x^2+1$, $F(0)=4$

[Sol] $F(x)=\int(3x^2+1)dx=x^3+x+C$

$F(0)=C=4$

Therefore, $\boldsymbol{F(x)=x^3+x+4}$

L 115a KUMON

Indefinite and Definite Integrals

Ex.

Given that $f(x)$ has a maximum value of 5 at $x=2$, and $f'(x)=-2x+a$, find $f(x)$.

[Sol] $f(x)=\int(-2x+a)dx$

$$=-x^2+ax+C$$

$$=-\left(x-\frac{a}{2}\right)^2+\frac{a^2}{4}+C$$

Therefore, $f(x)$ is a quadratic function with vertex $\left(\frac{a}{2},\ \frac{a^2}{4}+C\right)$.

From $\frac{a}{2}=2$, $\frac{a^2}{4}+C=5$

The x-coordinate of the vertex is 2.
The y-coordinate of the vertex is 5.

$a=4$, $C=1$

Therefore, $f(x)=-x^2+4x+1$

1. Given that $f(x)$ has a maximum value of 3 at $x=2$, and $f'(x)=-4x+a$, find $f(x)$.

[Sol] $f(x)=\int(-4x+a)dx$

$$=-2x^2+ax+C$$

$$=-2\left(x-\frac{a}{4}\right)^2+\frac{a^2}{8}+C$$

From $\frac{a}{4}=2$, $\frac{a^2}{8}+C=3$

$a=8$, $C=-5$

Therefore, $\boldsymbol{f(x)=-2x^2+8x-5}$

L115b

2. Given that $f(x)$ has a minimum value of -1 at $x=1$, and $f'(x)=x+a$, find $f(x)$.

[Sol] $f(x)=\int(x+a)dx$

$$=\frac{1}{2}x^2+ax+C$$

$$=\frac{1}{2}(x+a)^2-\frac{1}{2}a^2+C$$

From $a=-1$, $-\frac{1}{2}a^2+C=-1$

$$C=-\frac{1}{2}$$

Therefore, $\boldsymbol{f(x)=\frac{1}{2}x^2-x-\frac{1}{2}}$

L 116a KUMON

Indefinite and Definite Integrals

Ex.

The curve $y=f(x)$ passes through point $(1,\ 1)$. Find $f(x)$ when the gradient of the tangent at any point $(x,\ y)$ on the curve is $3x^2+2$.

[Sol] From $f'(x)=3x^2+2$,

$$f(x)=\int(3x^2+2)dx=x^3+2x+C$$

From $f(1)=1+2+C=1$, ☜ Substituting point $(1,\ 1)$ into $y=f(x)$.

$C=-2$

Therefore, $f(x)=x^3+2x-2$

1. The curve $y=f(x)$ passes through point $(1,\ 2)$. Find $f(x)$ when the gradient of the tangent at any point $(x,\ y)$ on the curve is $3x^2$.

[Sol] From $f'(x)=3x^2$,

$$f(x)=\int 3x^2dx=x^3+C$$

From $f(1)=1+C=2$,

$C=1$

Therefore, $\boldsymbol{f(x)=x^3+1}$

Note: The gradient of a curve at any point is equal to the gradient of the tangent at that point. To find the equation of the curve from a formula for the gradient, integrate.

2. The curve $y=f(x)$ crosses the y-axis at point $(0,\ 7)$. Find $f(x)$ when the gradient of the tangent at any point $(x,\ y)$ on the curve is $4x-8$.

[Sol] From $f'(x)=4x-8$,

$$f(x)=\int(4x-8)dx=2x^2-8x+C$$

$$f(0)=C=7$$

Therefore, $\boldsymbol{f(x)=2x^2-8x+7}$

3. The curve $y=f(x)$ passes through point $(2,\ -1)$. Find $f(x)$ when the gradient of the tangent at any point $(x,\ y)$ on the curve is $-3x^2+2x-1$.

[Sol] From $f'(x)=-3x^2+2x-1$,

$$f(x)=\int(-3x^2+2x-1)dx$$

$$=-x^3+x^2-x+C$$

From $f(2)=-8+4-2+C=-1$,

$$C=5$$

Therefore, $\boldsymbol{f(x)=-x^3+x^2-x+5}$

L 117a KUMON

Indefinite and Definite Integrals

1. Given that the indefinite integral of the function $f(x)=2x+1$ is $F(x)=\int(2x+1)dx=x^2+x+C$, find the value of $F(2)-F(1)$ when the constant of integration C has the following values.

Ex.

When $C=0$

[Sol] $F(x)=x^2+x$ ☜ Substituting 0 into C.

$F(2)-F(1)=(2^2+2)-(1^2+1)=4$

(1) When $C=1$

[Sol] $F(x)=\boldsymbol{x^2+x+1}$

$F(2)-F(1)=(2^2+2+1)-(1^2+1+1)=\mathbf{4}$

(2) When $C=2$

[Sol] $F(x)=x^2+x+2$

$F(2)-F(1)=(2^2+2+2)-(1^2+1+2)=\mathbf{4}$

(3) When $C=-1$

[Sol] $F(x)=x^2+x-1$

$F(2)-F(1)=(2^2+2-1)-(1^2+1-1)=\mathbf{4}$

➢ From questions (1), (2) and (3), we can see that $F(2)-F(1)$, for the indefinite integral $F(x)=x^2+x+C$ of the function $f(x)=2x+1$, has the same value regardless of the constant of integration C.

Generally, when the indefinite integral of $f(x)$ is $F(x)$, then $F(b)-F(a)$ is called the ***definite integral*** of $f(x)$ from a to b, and is written as $\int_a^b f(x)dx$.

L117b

Given that the indefinite integral of $f(x)$ is $F(x)$, then $F(b)-F(a)$ can be written as $\Big[F(x)\Big]_a^b$ and the following definition can be made:

Definition of Definite Integral

When the indefinite integral of $f(x)$ is $F(x)$,

$$\int_a^b f(x)dx=\Big[F(x)\Big]_a^b=F(b)-F(a)$$

2. Evaluate the following definite integrals.

Ex.

$$\int_1^2 (2x+1)dx=\Big[x^2+x\Big]_1^2=6-2=4$$

☜ $F(2)=2^2+2=6$
$F(1)=1^2+1=2$

(1) $\displaystyle\int_1^3 (2x+1)dx=\Big[x^2+x\Big]_1^3=12-2=\mathbf{10}$

(2) $\displaystyle\int_0^1 (-2x-1)dx=\Big[-x^2-x\Big]_0^1=-2-0=\mathbf{-2}$

(3) $\displaystyle\int_{-2}^1 (x+1)dx=\Big[\frac{1}{2}x^2+x\Big]_{-2}^1=\frac{3}{2}-0=\mathbf{\frac{3}{2}}$

Evaluating the definite integral $\int_a^b f(x)dx$ is called ***integrating from a to b***.

L 118a KUMON

Indefinite and Definite Integrals

Definition of Definite Integral

When the indefinite integral of $f(x)$ is $F(x)$,

$$\int_a^b f(x)dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

Evaluate the following definite integrals.

(1) $\displaystyle\int_0^1 x^2 dx = \left[\frac{1}{3}x^3\right]_0^1 = \mathbf{\frac{1}{3}}$

(2) $\displaystyle\int_0^1 (-x^2) dx = \left[-\frac{1}{3}x^3\right]_0^1 = \mathbf{-\frac{1}{3}}$

(3) $\displaystyle\int_0^{-1} x^2 dx = \left[\frac{1}{3}x^3\right]_0^{-1} = \mathbf{-\frac{1}{3}}$

(4) $\displaystyle\int_{-1}^1 x^2 dx = \left[\frac{1}{3}x^3\right]_{-1}^1 = \frac{1}{3} - \left(-\frac{1}{3}\right) = \mathbf{\frac{2}{3}}$

(5) $\displaystyle\int_a^a t^2 dt = \left[\frac{1}{3}t^3\right]_a^a = \mathbf{0}$

(6) $\displaystyle\int_1^a 2x dx = \Big[x^2\Big]_1^a = \boldsymbol{a^2 - 1}$

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（7） $\int_0^1 (x^2+2)dx = \left[\frac{1}{3}x^3+2x\right]_0^1 = \frac{1}{3}+2 = \mathbf{\frac{7}{3}}$

（8） $\int_1^2 (x^2+2x)dx = \left[\frac{1}{3}x^3+x^2\right]_1^2 = \left(\frac{8}{3}+4\right)-\left(\frac{1}{3}+1\right) = \mathbf{\frac{16}{3}}$

（9） $\int_1^2 (t^2+3)dt = \left[\frac{1}{3}t^3+3t\right]_1^2 = \left(\frac{8}{3}+6\right)-\left(\frac{1}{3}+3\right) = \mathbf{\frac{16}{3}}$

（10） $\int_{-1}^3 (4x-x^2)dx = \left[2x^2-\frac{1}{3}x^3\right]_{-1}^3 = (18-9)-\left(2+\frac{1}{3}\right) = \mathbf{\frac{20}{3}}$

（11） $\int_0^3 (x^2-2x+5)dx = \left[\frac{1}{3}x^3-x^2+5x\right]_0^3 = 9-9+15 = \mathbf{15}$

L 119a KUMON

Indefinite and Definite Integrals

1. Evaluate the following definite integrals.

(1) $\int_{-2}^{1}(x-1)(x+1)dx = \int_{-2}^{1}(x^2-1)dx$

$$= \left[\frac{1}{3}x^3 - x\right]_{-2}^{1} = \left(\frac{1}{3}-1\right)-\left(-\frac{8}{3}+2\right)$$

$$= \mathbf{0}$$

(2) $\int_{-1}^{3}x(3x-2)dx = \int_{-1}^{3}(3x^2-2x)dx = \left[x^3-x^2\right]_{-1}^{3}$

$$= (27-9)-(-1-1) = \mathbf{20}$$

(3) $\int_{1}^{3}(x+2)^2dx = \int_{1}^{3}(x^2+4x+4)dx = \left[\frac{1}{3}x^3+2x^2+4x\right]_{1}^{3}$

$$= (9+18+12)-\left(\frac{1}{3}+2+4\right)$$

$$= \mathbf{\frac{98}{3}}$$

(4) $\int_{0}^{2}(t-2)^2dt = \int_{0}^{2}(t^2-4t+4)dt$

$$= \left[\frac{1}{3}t^3-2t^2+4t\right]_{0}^{2} = \frac{8}{3}-8+8 = \mathbf{\frac{8}{3}}$$

(5) $\int_{0}^{2}x^2(x+1)dx = \int_{0}^{2}(x^3+x^2)dx$

$$= \left[\frac{1}{4}x^4+\frac{1}{3}x^3\right]_{0}^{2} = 4+\frac{8}{3} = \mathbf{\frac{20}{3}}$$

L119b

2. Find the function $f(x)$ that satisfies the following conditions.

Ex.

$f'(x)=2x-1$, $\int_0^1 f(x)dx=\frac{5}{6}$

[Sol] $f(x)=\int(2x-1)dx=x^2-x+C$

$$\int_0^1 f(x)dx=\int_0^1(x^2-x+C)dx$$

$$=\left[\frac{1}{3}x^3-\frac{1}{2}x^2+Cx\right]_0^1$$

$$=\frac{1}{3}-\frac{1}{2}+C=\frac{5}{6}$$

Therefore, $C=1$

Thus, $f(x)=x^2-x+1$

(1) $f'(x)=4x+2$, $\int_0^1 f(x)dx=\frac{2}{3}$

[Sol] $f(x)=\int(4x+2)dx=2x^2+2x+C$

$$\int_0^1 f(x)dx=\int_0^1(2x^2+2x+C)dx$$

$$=\left[\frac{2}{3}x^3+x^2+Cx\right]_0^1$$

$$=\frac{2}{3}+1+C=\frac{2}{3}$$

Therefore, $C=-1$

Thus, $\boldsymbol{f(x)=2x^2+2x-1}$

L 120a KUMON

Indefinite and Definite Integrals

1. Find the indefinite integral of $3x^2+4x$ given that when $x=1$, the indefinite integral is -2.

[Sol] Let the indefinite integral be $F(x)$,

$$F(x)=\int(3x^2+4x)dx=x^3+2x^2+C$$

From $F(1)=1+2+C=-2$,

$$C=-5$$

Therefore, the indefinite integral is $\boldsymbol{x^3+2x^2-5}$.

2. The curve $y=f(x)$ passes through point $(-2,\ -5)$. Find $f(x)$ when the gradient of the tangent at any point $(x,\ y)$ on the curve is $6x^2-2x-3$.

[Sol] From $f'(x)=6x^2-2x-3$,

$$f(x)=\int(6x^2-2x-3)dx$$

$$=2x^3-x^2-3x+C$$

From $f(-2)=-16-4+6+C=-5$,

$$C=9$$

Therefore, $\boldsymbol{f(x)=2x^3-x^2-3x+9}$

L 120b

3. Evaluate the following definite integrals.

(1) $\int_1^2 (4x^3-6x^2)dx = \left[x^4-2x^3\right]_1^2 = (16-16)-(1-2)$

$= \mathbf{1}$

(2) $\int_1^4 (x-1)^2 dx = \int_1^4 (x^2-2x+1)dx$

$= \left[\frac{1}{3}x^3 - x^2 + x\right]_1^4$

$= \left(\frac{64}{3}-16+4\right)-\left(\frac{1}{3}-1+1\right)$

$= \mathbf{9}$

(3) $\int_{-a}^{a} (a^2-t^2)dt = \left[a^2 t - \frac{1}{3}t^3\right]_{-a}^{a}$

$= \left(a^3-\frac{1}{3}a^3\right)-\left(-a^3+\frac{1}{3}a^3\right)$

$= \mathbf{\frac{4}{3}a^3}$

L121a KUMON

Definite Integrals I

The Properties of Indefinite Integrals from L112 are also true for definite integrals.

Properties of Definite Integrals I

1. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$ (k is a constant.)

2. $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

3. $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

1. Evaluate the following definite integral, using two different methods.

$$I = \int_{-1}^{2}(x^2-3x+1)dx$$

(i) $$I = \left[\frac{1}{3}x^3 - \frac{3}{2}x^2 + x\right]_{-1}^{2}$$
$$= \left(\frac{8}{3}-6+2\right)-\left(-\frac{1}{3}-\frac{3}{2}-1\right)$$
$$= \mathbf{\frac{3}{2}}$$

(ii) $$I = \int_{-1}^{2}x^2dx - \int_{-1}^{2}3xdx + \int_{-1}^{2}dx$$
$$= \left[\frac{1}{3}x^3\right]_{-1}^{2} - \left[\frac{3}{2}x^2\right]_{-1}^{2} + \left[x\right]_{-1}^{2}$$
$$= \frac{1}{3}(8+1) - \frac{3}{2}(4-1) + 3$$
$$= \mathbf{\frac{3}{2}}$$

L121b

2. Evaluate the following definite integrals.

(1) $\displaystyle\int_{-1}^{2}(x^2-5x+4)dx=\left[\frac{1}{3}x^3\right]_{-1}^{2}-\left[\frac{5}{2}x^2\right]_{-1}^{2}+\left[\,4x\,\right]_{-1}^{2}$

$\displaystyle=\frac{1}{3}(8+1)-\frac{5}{2}(4-1)+4(2+1)$

$\displaystyle=\mathbf{\frac{15}{2}}$

(2) $\displaystyle\int_{-1}^{3}(-x^2+5x-3)dx=-\left[\frac{1}{3}x^3\right]_{-1}^{3}+\left[\frac{5}{2}x^2\right]_{-1}^{3}-\left[\,3x\,\right]_{-1}^{3}$

$\displaystyle=-\frac{1}{3}(27+1)+\frac{5}{2}(9-1)-3(3+1)$

$\displaystyle=\mathbf{-\frac{4}{3}}$

(3) $\displaystyle\int_{0}^{3}(2x^2-2x-5)dx=\left[\frac{2}{3}x^3\right]_{0}^{3}-\left[\,x^2\,\right]_{0}^{3}-\left[\,5x\,\right]_{0}^{3}$

$=18-9-15$

$=\mathbf{-6}$

L 122a KUMON

Definite Integrals I

Evaluate the following definite integrals.

(1) $$\int_{-1}^{2}(3x^2-4x+3)dx=\Big[x^3-2x^2+3x\Big]_{-1}^{2}$$
$$=(8-8+6)-(-1-2-3)$$
$$=\mathbf{12}$$

(2) $$\int_{-1}^{2}(2x^2+3x-5)dx=\Big[\frac{2}{3}x^3+\frac{3}{2}x^2-5x\Big]_{-1}^{2}$$
$$=\left(\frac{16}{3}+6-10\right)-\left(-\frac{2}{3}+\frac{3}{2}+5\right)$$
$$=\boldsymbol{-\frac{9}{2}}$$

(3) $$\int_{2}^{-1}(x^3-2x+1)dx=\Big[\frac{1}{4}x^4-x^2+x\Big]_{2}^{-1}$$
$$=\left(\frac{1}{4}-1-1\right)-(4-4+2)$$
$$=\boldsymbol{-\frac{15}{4}}$$

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(4) $\int_{-1}^{3}(x+1)(3-x)dx=\int_{-1}^{3}(-x^2+2x+3)dx=\left[-\frac{1}{3}x^3+x^2+3x\right]_{-1}^{3}$

$$=(-9+9+9)-\left(\frac{1}{3}+1-3\right)$$

$$=\mathbf{\frac{32}{3}}$$

(5) $\int_{-2}^{1}(y+2)^2dy=\int_{-2}^{1}(y^2+4y+4)dy=\left[\frac{1}{3}y^3+2y^2+4y\right]_{-2}^{1}$

$$=\left(\frac{1}{3}+2+4\right)-\left(-\frac{8}{3}+8-8\right)$$

$$=\mathbf{9}$$

(6) $\int_{-2}^{2}(x+1)(x-2)dx=\int_{-2}^{2}(x^2-x-2)dx=\left[\frac{1}{3}x^3-\frac{1}{2}x^2-2x\right]_{-2}^{2}$

$$=\left(\frac{8}{3}-2-4\right)-\left(-\frac{8}{3}-2+4\right)$$

$$=\mathbf{-\frac{8}{3}}$$

(7) $\int_{0}^{1}(x+1)^2dx-\int_{0}^{1}(x-1)^2dx$

☞ Since the upper and lower limits of both integrals are the same, try to find a simpler way to evaluate. Use property **3.** from L I 2 I a, in reverse.

$$=\int_{0}^{1}[(x+1)^2-(x-1)^2]dx$$

$$=\int_{0}^{1}4xdx$$

$$=\left[2x^2\right]_{0}^{1}$$

$$=\mathbf{2}$$

L 123a KUMON

Definite Integrals I

1. Evaluate the following definite integrals.

(1) $\int_0^3 dx = \Big[x \Big]_0^3 = \mathbf{3}$

(2) $\int_{-3}^3 dx = \Big[x \Big]_{-3}^3 = 3-(-3) = \mathbf{6}$

(3) $\int_0^3 x\,dx = \left[\frac{1}{2}x^2\right]_0^3 = \mathbf{\frac{9}{2}}$

(4) $\int_{-3}^3 x\,dx = \left[\frac{1}{2}x^2\right]_{-3}^3 = \frac{9}{2} - \frac{9}{2} = \mathbf{0}$

(5) $\int_0^3 x^2 dx = \left[\frac{1}{3}x^3\right]_0^3 = \mathbf{9}$

(6) $\int_{-3}^3 x^2 dx = \left[\frac{1}{3}x^3\right]_{-3}^3 = 9-(-9) = \mathbf{18}$

(7) $\int_0^3 x^3 dx = \left[\frac{1}{4}x^4\right]_0^3 = \mathbf{\frac{81}{4}}$

(8) $\int_{-3}^3 x^3 dx = \left[\frac{1}{4}x^4\right]_{-3}^3 = \frac{81}{4} - \frac{81}{4} = \mathbf{0}$

2. Fill in the blank boxes.

(i) From (4) and (8) above, when n is 1 or 3, $\int_{-3}^3 x^n dx = \boxed{\mathbf{0}}$.

(ii) From (1), (2), (5) and (6) above, when n is 0 or 2,

$$\int_{-3}^3 x^n dx = 2\int_{\boxed{\mathbf{0}}}^3 x^n dx.$$

Properties of Definite Integrals II

4. $\int_{-a}^{a} x^n dx = \begin{cases} 0 & (\text{When } n \text{ is } 1, 3, 5, \ldots) \\ 2\int_0^a x^n dx & (\text{When } n \text{ is } 0, 2, 4, \ldots) \end{cases}$

3. Using the property of definite integrals above, evaluate the following definite integrals.

(1) $$\int_{-1}^{1}(x^3+3x^2-2x+1)dx = \int_{-1}^{1}x^3dx + 3\int_{-1}^{1}x^2dx - 2\int_{-1}^{1}xdx + \int_{-1}^{1}dx$$
$$= 6\int_0^1 x^2dx + \boxed{\mathbf{2\int_0^1 dx}}$$
$$= 6\left[\frac{1}{3}x^3\right]_0^1 + 2\Big[\,x\,\Big]_0^1 = 2+2 = \mathbf{4}$$

(2) $$\int_{-3}^{3}(5x^3+2x^2+6x-1)dx = 5\int_{-3}^{3}x^3dx + 2\int_{-3}^{3}x^2dx + 6\int_{-3}^{3}x\,dx - \int_{-3}^{3}dx$$
$$= 4\int_0^3 x^2dx - 2\int_0^3 dx$$
$$= 4\left[\frac{1}{3}x^3\right]_0^3 - 2\Big[\,x\,\Big]_0^3 = 36-6 = \mathbf{30}$$

(3) $$\int_{-1}^{1}(x^3-1)dx = \int_{-1}^{1}x^3dx - \int_{-1}^{1}dx$$
$$= -2\int_0^1 dx = -2\Big[\,x\,\Big]_0^1 = \mathbf{-2}$$

(4) $$\int_{-a}^{a}(Ax^4+Bx^3+Cx^2+Dx+E)dx$$
$$= A\int_{-a}^{a}x^4dx + B\int_{-a}^{a}x^3dx + C\int_{-a}^{a}x^2dx + D\int_{-a}^{a}x\,dx + E\int_{-a}^{a}dx$$
$$= 2A\int_0^a x^4dx + 2C\int_0^a x^2dx + 2E\int_0^a dx$$
$$= 2A\left[\frac{1}{5}x^5\right]_0^a + 2C\left[\frac{1}{3}x^3\right]_0^a + 2E\Big[\,x\,\Big]_0^a = \boldsymbol{\frac{2}{5}Aa^5 + \frac{2}{3}Ca^3 + 2Ea}$$

L 124a KUMON

Definite Integrals I

1. Evaluate the following definite integrals.

(1) $\displaystyle\int_1^3 (x-1)(x-3)dx = \int_1^3 (x^2-4x+3)dx$

$$= \left[\frac{1}{3}x^3 - 2x^2 + 3x\right]_1^3$$

$$= (9-18+9) - \left(\frac{1}{3} - 2 + 3\right)$$

$$= -\boldsymbol{\frac{4}{3}}$$

(2) $\displaystyle\int_\alpha^\beta (x-\alpha)(x-\beta)dx$

$$= \int_\alpha^\beta [x^2 - (\alpha+\beta)x + \alpha\beta]dx$$

$$= \left[\frac{1}{3}x^3\right]_\alpha^\beta - (\alpha+\beta)\left[\frac{1}{2}x^2\right]_\alpha^\beta + \alpha\beta\left[\ x\ \right]_\alpha^\beta$$

$$= \frac{1}{3}(\beta^3-\alpha^3) - \frac{1}{2}(\ \boldsymbol{\alpha+\beta}\ )(\beta^2-\alpha^2) + \boxed{\boldsymbol{\alpha\beta}}(\beta-\alpha)$$

$$= \frac{1}{6}(\beta-\alpha)[2(\beta^2+\alpha\beta+\alpha^2) - 3(\ \boldsymbol{\beta+\alpha}\ )^2 + 6\boxed{\boldsymbol{\alpha\beta}}\,]$$

$$= \frac{1}{6}(\beta-\alpha)(\ \boldsymbol{-\beta^2+2\alpha\beta-\alpha^2}\ )$$

$$= -\frac{1}{6}(\beta-\alpha)^3$$

2. Fill in the blank boxes.

Compare (1) and (2) above.
Using the result of (2), and putting $\alpha = 1$ and $\beta = 3$, we can find the answer to (1) as follows:

$$\int_1^3 (x-1)(x-3)dx = -\frac{1}{6}(\boxed{\mathbf{3}} - \boxed{\mathbf{1}})^3 = -\frac{4}{3}$$

Properties of Definite Integrals III

5. $\int_{\alpha}^{\beta}(x-\alpha)(x-\beta)dx=-\frac{1}{6}(\beta-\alpha)^3$

3. Using the property of definite integrals above, evaluate the following definite integrals.

(1) $\int_{-2}^{1}(x+2)(x-1)dx=-\frac{1}{6}[1-(-2)]^3=\mathbf{-\frac{9}{2}}$

(2) $\int_{1}^{3}(x^2-4x+3)dx=\int_{1}^{3}(x-1)(x-3)dx=-\frac{1}{6}(3-1)^3=\mathbf{-\frac{4}{3}}$

(3) $\int_{-3}^{\frac{1}{2}}(2x-1)(x+3)dx=\boxed{\mathbf{2}}\int_{-3}^{\frac{1}{2}}\left(x-\boxed{\mathbf{\frac{1}{2}}}\right)(x+3)dx$

$=2\cdot\left(-\frac{1}{6}\right)\left[\frac{1}{2}-(-3)\right]^3=\mathbf{-\frac{343}{24}}$

(4) $\int_{-\frac{1}{2}}^{3}(x-3)(2x+1)dx=2\int_{-\frac{1}{2}}^{3}(x-3)\left(x+\frac{1}{2}\right)dx$

$=2\cdot\left(-\frac{1}{6}\right)\left[3-\left(-\frac{1}{2}\right)\right]^3=\mathbf{-\frac{343}{24}}$

L 125a KUMON

Definite Integrals I

Evaluate the following definite integrals. Assume that α and β are the two distinct real roots of the integrand, and $\alpha<\beta$.

Ex.

$$\int_{\alpha}^{\beta}(x^2-2x-1)dx$$

[Sol] $\alpha+\beta=2,\ \alpha\beta=-1$ ☜ From the root-coefficient relationship.

From $(\beta-\alpha)^2=(\alpha+\beta)^2-4\alpha\beta=8$,

$\beta-\alpha=2\sqrt{2}$ ☜ Since $\alpha<\beta$, $-2\sqrt{2}$ is an extraneous solution.

Therefore,

$$\int_{\alpha}^{\beta}(x^2-2x-1)dx=-\frac{1}{6}(\beta-\alpha)^3=-\frac{1}{6}(2\sqrt{2})^3=-\frac{8}{3}\sqrt{2}$$

(1) $\displaystyle\int_{\alpha}^{\beta}(x^2-3x+1)dx$

[Sol] $\alpha+\beta=3,\ \alpha\beta=1$

From $(\beta-\alpha)^2=(\alpha+\beta)^2-4\alpha\beta=5$,

$\beta-\alpha=\sqrt{5}$

Therefore,

$$\int_{\alpha}^{\beta}(x^2-3x+1)dx=-\frac{1}{6}(\beta-\alpha)^3=-\frac{1}{6}(\sqrt{5})^3=\boldsymbol{-\frac{5}{6}\sqrt{5}}$$

In any integral, the function being integrated is called the ***integrand***.
In the example above, the integrand is x^2-2x-1.

L125b

(2) $\int_{\alpha}^{\beta}(x^2+2x-5)dx$

[Sol] $\alpha+\beta=-2$, $\alpha\beta=-5$

From $(\beta-\alpha)^2=(\alpha+\beta)^2-4\alpha\beta=24$,

$\beta-\alpha=2\sqrt{6}$

Therefore,

$$\int_{\alpha}^{\beta}(x^2+2x-5)dx=-\frac{1}{6}(\beta-\alpha)^3=-\frac{1}{6}(2\sqrt{6})^3=\mathbf{-8\sqrt{6}}$$

(3) $\int_{\alpha}^{\beta}(x^2-2ax+a^2-3)dx$

[Sol] $\alpha+\beta=2a$, $\alpha\beta=a^2-3$

From $(\beta-\alpha)^2=(\alpha+\beta)^2-4\alpha\beta=(2a)^2-4(a^2-3)=12$,

$\beta-\alpha=2\sqrt{3}$

Therefore,

$$\int_{\alpha}^{\beta}(x^2-2ax+a^2-3)dx=-\frac{1}{6}(\beta-\alpha)^3=-\frac{1}{6}(2\sqrt{3})^3=\mathbf{-4\sqrt{3}}$$

L 126a KUMON

Definite Integrals I

1. Evaluate the following definite integrals.

(1) $\int_0^2 (2x+3)dx = \Big[x^2+3x\Big]_0^2 = 4+6 = \mathbf{10}$

(2) $\int_2^0 (2x+3)dx = \Big[x^2+3x\Big]_2^0 = -(4+6) = \mathbf{-10}$

(3) $\int_{-1}^0 (x^2-2x+1)dx = \Big[\frac{1}{3}x^3-x^2+x\Big]_{-1}^0 = -\Big(-\frac{1}{3}-1-1\Big) = \mathbf{\frac{7}{3}}$

(4) $\int_0^3 (x^2-2x+1)dx = \Big[\frac{1}{3}x^3-x^2+x\Big]_0^3 = 9-9+3 = \mathbf{3}$

(5) $\int_{-1}^3 (x^2-2x+1)dx = \Big[\frac{1}{3}x^3-x^2+x\Big]_{-1}^3 = (9-9+3)-\Big(-\frac{1}{3}-1-1\Big)$

$= \mathbf{\frac{16}{3}}$

2. Fill in the blank boxes.

(i) From (1) and (2) above, $\int_2^0 (2x+3)dx = -\int_{\boxed{\mathbf{0}}}^{\boxed{\mathbf{2}}} (2x+3)dx$

(ii) From (3)～(5) above,

$$\int_{-1}^3 (x^2-2x+1)dx = \int_{-1}^{\boxed{\mathbf{0}}} (x^2-2x+1)dx + \int_{\boxed{\mathbf{0}}}^3 (x^2-2x+1)dx$$

Properties of Definite Integrals Ⅳ

6. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

7. $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$

3. Using the properties of definite integrals above, evaluate the following definite integrals.

(1) $\int_{-1}^{0}(2x^2-x)dx+\int_0^2(2x^2-x)dx$

$= \int_{\boxed{-1}}^{\boxed{2}}(\ \mathbf{2x^2-x}\ )dx$

$= \left[\frac{2}{3}x^3-\frac{1}{2}x^2\right]_{-1}^{2} = \left(\frac{16}{3}-2\right)-\left(-\frac{2}{3}-\frac{1}{2}\right)= \mathbf{\frac{9}{2}}$

(2) $\int_0^1(x^2-x+1)dx-\int_2^1(x^2-x+1)dx$

$= \int_0^1(x^2-x+1)dx+\int_1^2(x^2-x+1)dx$

$= \int_0^2(x^2-x+1)dx$

$= \left[\frac{1}{3}x^3-\frac{1}{2}x^2+x\right]_0^2 = \frac{8}{3}-2+2 = \mathbf{\frac{8}{3}}$

(3) $\int_0^2(x^2-4x+3)dx-\int_0^{-1}(x^2-4x+3)dx$

$= \int_0^2(x^2-4x+3)dx+\int_{-1}^0(x^2-4x+3)dx$

$= \int_{-1}^2(x^2-4x+3)dx$

$= \left[\frac{1}{3}x^3-2x^2+3x\right]_{-1}^2 = \left(\frac{8}{3}-8+6\right)-\left(-\frac{1}{3}-2-3\right)= \mathbf{6}$

L 127a KUMON

Definite Integrals I

1. Evaluate the following definite integrals.

Ex.

$$\int_0^2 |x^2-1|dx$$

[Sol] From $x^2-1=(x+1)(x-1)$,

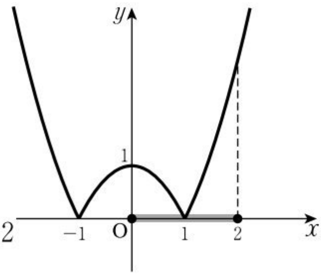
$$|x^2-1|=\begin{cases}-x^2+1 & (0 \leqslant x \leqslant 1) \quad x^2-1 \leqslant 0 \text{ when } 0 \leqslant x \leqslant 1 \\ x^2-1 & (1 \leqslant x \leqslant 2) \quad x^2-1 \geqslant 0 \text{ when } 1 \leqslant x \leqslant 2\end{cases}$$

$$\int_0^2 |x^2-1|dx$$

$$=\int_0^1(-x^2+1)dx+\int_1^2(x^2-1)dx$$

$$=\left[-\frac{1}{3}x^3+x\right]_0^1+\left[\frac{1}{3}x^3-x\right]_1^2$$

$$=\left(-\frac{1}{3}+1\right)+\left(\frac{8}{3}-2\right)-\left(\frac{1}{3}-1\right)=2$$



(1) $\int_0^2 |x^2+x-2|dx$

[Sol] From $x^2+x-2=\boldsymbol{(x+2)(x-1)}$,

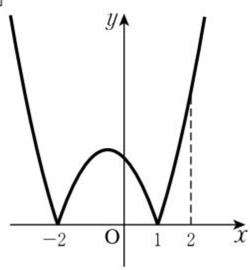
$$|x^2+x-2|=\begin{cases}\boxed{\boldsymbol{-x^2-x+2}} & (\boxed{\mathbf{0}} \leqslant x \leqslant \boxed{\mathbf{1}}) \\ \boxed{\boldsymbol{x^2+x-2}} & (\boxed{\mathbf{1}} \leqslant x \leqslant \boxed{\mathbf{2}})\end{cases}$$

$$\int_0^2 |x^2+x-2|dx$$

$$=\int_0^1(-x^2-x+2)dx+\int_1^2(x^2+x-2)dx$$

$$=\left[-\frac{1}{3}x^3-\frac{1}{2}x^2+2x\right]_0^1+\left[\frac{1}{3}x^3+\frac{1}{2}x^2-2x\right]_1^2$$

$$=\left(-\frac{1}{3}-\frac{1}{2}+2\right)+\left(\frac{8}{3}+2-4\right)-\left(\frac{1}{3}+\frac{1}{2}-2\right)=\mathbf{3}$$



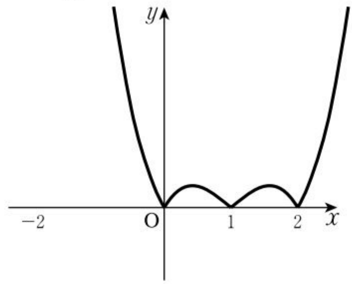
L127b

(2) $\displaystyle\int_{-2}^{1} |x(x-1)(x-2)|dx$

[Sol] $|x(x-1)(x-2)| = \begin{cases} -x(x-1)(x-2) & (-2 \leqq x \leqq 0) \\ x(x-1)(x-2) & (0 \leqq x \leqq 1) \end{cases}$

Therefore,

$$\int_{-2}^{1} |x(x-1)(x-2)|dx$$
$$= \int_{-2}^{0} [-x(x-1)(x-2)]dx + \int_{0}^{1} x(x-1)(x-2)dx$$
$$= -\int_{-2}^{0} (x^3-3x^2+2x)dx + \int_{0}^{1} (x^3-3x^2+2x)dx$$
$$= -\left[\frac{1}{4}x^4 - x^3 + x^2\right]_{-2}^{0} + \left[\frac{1}{4}x^4 - x^3 + x^2\right]_{0}^{1}$$
$$= (4+8+4) + \left(\frac{1}{4} - 1 + 1\right)$$
$$= \mathbf{\frac{65}{4}}$$



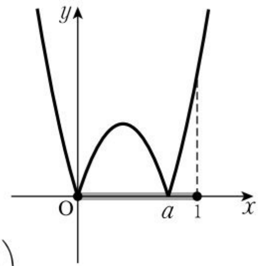
L 128a KUMON

Definite Integrals I

Ex.

Evaluate the integral $I=\int_0^1 |x(x-a)|dx$ when $0<a<1$.

[Sol] $|x(x-a)|=\begin{cases}-x(x-a) & (0\leqq x\leqq a)\\ x(x-a) & (a\leqq x\leqq 1)\end{cases}$



$$I=\int_0^a[-x(x-a)]dx+\int_a^1 x(x-a)dx$$

$$=-\left[\frac{1}{3}x^3-\frac{a}{2}x^2\right]_0^a+\left[\frac{1}{3}x^3-\frac{a}{2}x^2\right]_a^1$$

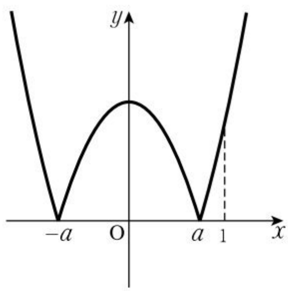
$$=\left(-\frac{a^3}{3}+\frac{a^3}{2}\right)+\left(\frac{1}{3}-\frac{a}{2}\right)-\left(\frac{a^3}{3}-\frac{a^3}{2}\right)$$

$$=\frac{a^3}{3}-\frac{a}{2}+\frac{1}{3}$$

1. Evaluate the integral $I=\int_0^1 |x^2-a^2|dx$ when $0<a<1$.

[Sol] From $x^2-a^2=(x+a)(x-a)$,

$$|x^2-a^2|=\begin{cases}-x^2+a^2 & (0\leqq x\leqq a)\\ x^2-a^2 & (a\leqq x\leqq 1)\end{cases}$$



$$I=\int_0^a(-x^2+a^2)dx+\int_a^1(x^2-a^2)dx$$

$$=\left[-\frac{1}{3}x^3+a^2x\right]_0^a+\left[\frac{1}{3}x^3-a^2x\right]_a^1$$

$$=\left(-\frac{a^3}{3}+a^3\right)+\left(\frac{1}{3}-a^2\right)-\left(\frac{a^3}{3}-a^3\right)$$

$$=\boldsymbol{\frac{4}{3}a^3-a^2+\frac{1}{3}}$$

L128b

2. Evaluate the integral $I=\int_0^2 |x(x-a)|dx$ when $a>0$.

(i) When $0<a<2$,

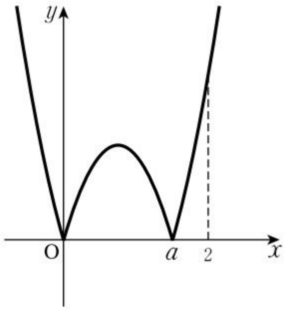
[Sol] $|x(x-a)|=\begin{cases} -x(x-a) & (0\leqslant x\leqslant a) \\ x(x-a) & (a\leqslant x\leqslant 2) \end{cases}$

$$I=\int_0^a [-x(x-a)]dx+\int_a^2 x(x-a)dx$$

$$=-\left[\frac{1}{3}x^3-\frac{a}{2}x^2\right]_0^a+\left[\frac{1}{3}x^3-\frac{a}{2}x^2\right]_a^2$$

$$=\left(-\frac{a^3}{3}+\frac{a^3}{2}\right)+\left(\frac{8}{3}-2a\right)-\left(\frac{a^3}{3}-\frac{a^3}{2}\right)$$

$$=\boldsymbol{\frac{a^3}{3}-2a+\frac{8}{3}}$$



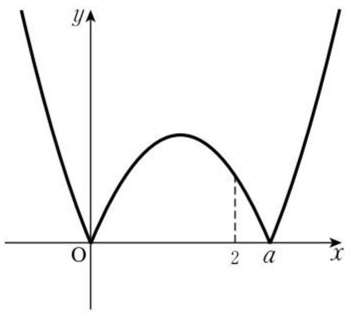
(ii) When $a\geqslant 2$,

[Sol] $|x(x-a)|=-x(x-a)$

$$I=\int_0^2 [-x(x-a)]dx$$

$$=-\left[\frac{1}{3}x^3-\frac{a}{2}x^2\right]_0^2$$

$$=\boldsymbol{-\frac{8}{3}+2a}$$



L 129a KUMON

Definite Integrals I

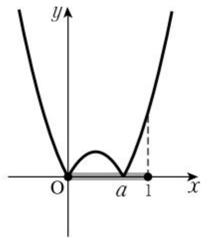
Ex.

Find the minimum value of $F(a)=\int_0^1|x(x-a)|dx$ when $a>0$.

[Sol]

(i) When $0<a<1$,

(i)



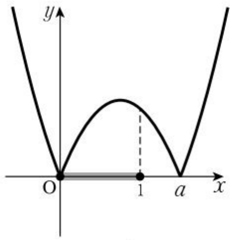
$$F(a)=\int_0^{\boxed{a}}[-x(x-a)]dx+\int_{\boxed{a}}^1 x(x-a)dx$$

$$=-\left[\frac{1}{3}x^3-\frac{a}{2}x^2\right]_0^{\boxed{a}}+\left[\frac{1}{3}x^3-\frac{a}{2}x^2\right]_{\boxed{a}}^1$$

$$=\left(-\frac{a^3}{3}+\frac{a^3}{2}\right)+\left(\frac{1}{3}-\frac{a}{2}\right)-\left(\frac{a^3}{3}-\frac{a^3}{2}\right)$$

$$=\frac{a^3}{3}-\frac{a}{2}+\frac{1}{3}$$

(ii)



$$F'(a)=a^2-\frac{1}{2}=\left(a+\frac{\sqrt{2}}{2}\right)\left(a-\frac{\sqrt{2}}{2}\right)$$

a	0	$\cdots$	$\frac{\sqrt{2}}{2}$	$\cdots$	1
$F'(a)$		$-$	0	$+$	
$F(a)$		$\searrow$	relative minimum	$\nearrow$	

Therefore, when $0<a<1$, the minimum value is $F\left(\frac{\sqrt{2}}{2}\right)=\boxed{\mathbf{\frac{2-\sqrt{2}}{6}}}$.

(ii) When $a\geqq 1$,

$$F(a)=-\int_0^1 x(x-a)dx=-\left[\frac{1}{3}x^3-\frac{a}{2}x^2\right]_0^1$$

$$=-\frac{1}{3}+\frac{a}{2}$$

☜ Considering the range of values $a\geqq 1$, this expression is smallest when $a=1$.

Therefore, when $a\geqq 1$, the minimum value is $F(1)=\boxed{\mathbf{\frac{1}{6}}}$.

From (i) and (ii), the overall minimum value is $\boxed{\mathbf{\frac{2-\sqrt{2}}{6}}}$, when $a=\frac{\sqrt{2}}{2}$.

Answers: in order a, a, a, a, $\frac{2-\sqrt{2}}{6}$, $\frac{1}{6}$, $\frac{2-\sqrt{2}}{6}$

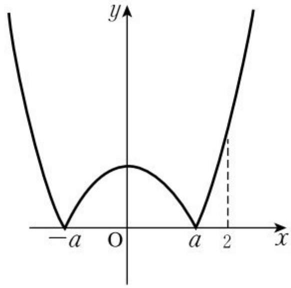
L 129b

1. Find the minimum value of $F(a)=\int_0^2 |x^2-a^2|dx$ when $a>0$.

[Sol]

(i) When $0<a<2$,

$$F(a)=\int_0^a [-(x^2-a^2)]dx+\int_a^2 (x^2-a^2)dx$$
$$=-\left[\frac{1}{3}x^3-a^2x\right]_0^a+\left[\frac{1}{3}x^3-a^2x\right]_a^2$$
$$=\left(-\frac{a^3}{3}+a^3\right)+\left(\frac{8}{3}-2a^2\right)-\left(\frac{a^3}{3}-a^3\right)$$
$$=\frac{4}{3}a^3-2a^2+\frac{8}{3}$$



$F'(a)=4a^2-4a=4a(a-1)$

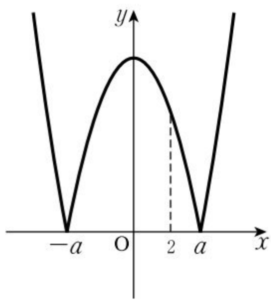
a	0	$\cdots$	1	$\cdots$	2
$F'(a)$		$-$	0	$+$	
$F(a)$		↘	relative minimum	↗	

Therefore, when $0<a<2$, the minimum value is

$$F(1)=\frac{4}{3}-2+\frac{8}{3}=2.$$

(ii) When $a\geqq 2$,

$$F(a)=\int_0^2 [-(x^2-a^2)]dx$$
$$=-\left[\frac{1}{3}x^3-a^2x\right]_0^2$$
$$=-\frac{8}{3}+2a^2$$



Therefore, when $a\geqq 2$, the minimum value is

$$F(2)=-\frac{8}{3}+8=\frac{16}{3}.$$

From (i) and (ii), **the overall minimum value is 2, when $a=1$.**

L 130a KUMON

Definite Integrals I

1. Evaluate the following definite integrals.

(1) $\int_{-2}^{1}(-x^2-x+2)dx$

$=\left[-\frac{1}{3}x^3-\frac{1}{2}x^2+2x\right]_{-2}^{1}$

$=\left(-\frac{1}{3}-\frac{1}{2}+2\right)-\left(\frac{8}{3}-2-4\right)=\boldsymbol{\frac{9}{2}}$

(2) $\int_{-2}^{2}(4x^3-3x^2+2x-1)dx$

$=4\int_{-2}^{2}x^3dx-3\int_{-2}^{2}x^2dx+2\int_{-2}^{2}xdx-\int_{-2}^{2}dx$

$=-6\int_{0}^{2}x^2dx-2\int_{0}^{2}dx$

$=-6\left[\frac{1}{3}x^3\right]_0^2-2\left[\,x\,\right]_0^2=-16-4=\mathbf{-20}$

(3) $\int_{1}^{3}(x^3+2x-1)dx+\int_{3}^{1}x^2(x-1)dx$

$=\int_{1}^{3}(x^3+2x-1)dx-\int_{1}^{3}(x^3-x^2)dx$

$=\int_{1}^{3}(x^2+2x-1)dx$

$=\left[\frac{1}{3}x^3+x^2-x\right]_1^3=(9+9-3)-\left(\frac{1}{3}+1-1\right)=\boldsymbol{\frac{44}{3}}$

L130b

2. Evaluate the integral $\int_{\alpha}^{\beta}(x^2-4x+2)dx$. Assume that α and β are the 2 distinct real roots of the integrand, and $\alpha<\beta$.

[Sol] $\alpha+\beta=4,\ \alpha\beta=2$

From $(\beta-\alpha)^2=(\alpha+\beta)^2-4\alpha\beta=16-8=8$,

$\beta-\alpha=2\sqrt{2}$

Therefore,

$$\int_{\alpha}^{\beta}(x^2-4x+2)dx=-\frac{1}{6}(\beta-\alpha)^3=-\frac{1}{6}\times(2\sqrt{2})^3=-\mathbf{\frac{8}{3}\sqrt{2}}$$

3. Evaluate the integral $I=\int_{-2}^{1}|x^2-x-2|dx$.

[Sol] From $x^2-x-2=(x-2)(x+1)$,

$$|x^2-x-2|=\begin{cases} x^2-x-2 & (-2\leqslant x\leqslant -1) \\ -x^2+x+2 & (-1\leqslant x\leqslant 1) \end{cases}$$

Therefore,

$$I=\int_{-2}^{-1}(x^2-x-2)dx+\int_{-1}^{1}(-x^2+x+2)dx$$

$$=\left[\frac{1}{3}x^3-\frac{1}{2}x^2-2x\right]_{-2}^{-1}+2\left[-\frac{1}{3}x^3+2x\right]_{0}^{1}$$

$$=\left(-\frac{1}{3}-\frac{1}{2}+2\right)-\left(-\frac{8}{3}-2+4\right)+2\left(-\frac{1}{3}+2\right)$$

$$=\mathbf{\frac{31}{6}}$$

L 131 a KUMON

Definite Integrals II

1. Each of the following definite integrals represents a function of x. Differentiate each one with respect to x.

Ex.

$$\int_0^x (t^3-t)\,dt$$

[Sol] $\displaystyle\int_0^x (t^3-t)\,dt = \left[\frac{1}{4}t^4-\frac{1}{2}t^2\right]_0^x = \frac{1}{4}x^4-\frac{1}{2}x^2$

$$\frac{d}{dx}\left(\frac{1}{4}x^4-\frac{1}{2}x^2\right)=x^3-x$$

Note: $\dfrac{d}{dx}\Big(f(x)\Big)$ is used to express: differentiating the function $f(x)$ with respect to x.

(1) $\displaystyle\int_1^x (t^3-t)dt = \left[\frac{1}{4}t^4-\frac{1}{2}t^2\right]_1^x = \left(\frac{1}{4}x^4-\frac{1}{2}x^2\right)-\left(\frac{1}{4}-\frac{1}{2}\right)$

$$=\frac{1}{4}x^4-\frac{1}{2}x^2+\frac{1}{4}$$

$$\frac{d}{dx}\left(\frac{1}{4}x^4-\frac{1}{2}x^2+\frac{1}{4}\right)=\boldsymbol{x^3-x}$$

(2) $\displaystyle\int_a^x (3t^2-4t+1)dt = \Big[t^3-2t^2+t\Big]_a^x = (x^3-2x^2+x)-(a^3-2a^2+a)$

$$=x^3-2x^2+x-a^3+2a^2-a$$

$$\frac{d}{dx}(x^3-2x^2+x-a^3+2a^2-a)=\boldsymbol{3x^2-4x+1}$$

(3) $\displaystyle\int_a^x (t^2-x^2)dt = \left[\frac{1}{3}t^3-x^2t\right]_a^x = \left(\frac{1}{3}x^3-x^3\right)-\left(\frac{1}{3}a^3-x^2a\right)$

$$=-\frac{2}{3}x^3+ax^2-\frac{1}{3}a^3$$

$$\frac{d}{dx}\left(-\frac{2}{3}x^3+ax^2-\frac{1}{3}a^3\right)=\boldsymbol{-2x^2+2ax}$$

Note: Question (3) is an exception to the formula given on side b. The formula does not apply here, as the integrand is a function of both t and x.

L131b

If a is a constant, $f(x)$ can be found by differentiating $\int_a^x f(t)dt$ as a function of x, with respect to x.

If $F(t)$ is the indefinite integral of $f(t)$,

$$\int_a^x f(t)dt = F(x) - F(a)$$

Differentiating both sides with respect to x,

$$\frac{d}{dx}\int_a^x f(t)dt = \frac{d}{dx}[F(x) - F(a)] = F'(x) = f(x)$$

☜ $\frac{d}{dx}[F(a)]$ is zero, as $F(a)$ is constant.

$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \qquad (a \text{ is a constant.})}$$

2. Differentiate the following functions.

(1) $f(x) = \int_x^1 (2t-1)dt = -\int_1^x (2t-1)dt$

$\frac{d}{dx}f(x) = -(2x-1) = \boldsymbol{-2x+1}$

(2) $f(x) = \int_x^a (t^2+2t+3)dt = -\int_a^x (t^2+2t+3)dt$

$\frac{d}{dx}f(x) = -(x^2+2x+3) = \boldsymbol{-x^2-2x-3}$

(3) $f(x) = \int_x^a (3t^2-4t+1)dt = -\int_a^x (3t^2-4t+1)dt$

$\frac{d}{dx}f(x) = -(3x^2-4x+1) = \boldsymbol{-3x^2+4x-1}$

L 132a KUMON

Definite Integrals II

Find the relative maximum value and the relative minimum value of each function.

Ex.

$F(x)=\int_{-1}^{x} 3t(t-2)dt$

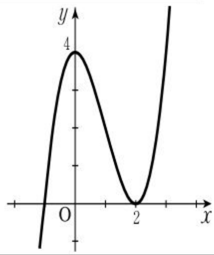
[Sol] $F(x)=\int_{-1}^{x}(3t^2-6t)dt$

$=\left[t^3-3t^2\right]_{-1}^{x}=x^3-3x^2+4$

Thus, $F'(x)=3x(x-2)$

We can also find this result from $\frac{d}{dx}\int_a^x f(t)dt=f(x)$

x	$\cdots$	0	$\cdots$	2	$\cdots$
$F'(x)$	$+$	0	$-$	0	$+$
$F(x)$	$\nearrow$	4	$\searrow$	0	$\nearrow$



The relative maximum value is 4, at $x=0$.

The relative minimum value is 0, at $x=2$.

(1) $F(x)=\int_{0}^{x} 3(t+3)(t-1)dt$

[Sol] $F(x)=\int_{0}^{x}(3t^2+6t-9)dt$

$=\left[t^3+3t^2-9t\right]_{0}^{x}=x^3+3x^2-9x$

Thus, $F'(x)=3(x+3)(x-1)$

Alternatively, just apply the formula on L131b.

x	$\cdots$	-3	$\cdots$	1	$\cdots$
$F'(x)$	$+$	0	$-$	0	$+$
$F(x)$	$\nearrow$	27	$\searrow$	-5	$\nearrow$

Therefore: **The relative maximum value is 27, at $x=-3$.**

The relative minimum value is -5, at $x=1$.

L I 32b

(2) $F(x)=\int_0^x t(t-1)(t-2)dt$

[Sol] $F(x)=\int_0^x (t^3-3t^2+2t)dt$

$$=\left[\frac{1}{4}t^4-t^3+t^2\right]_0^x=\frac{1}{4}x^4-x^3+x^2$$

Thus, $F'(x)=x(x-1)(x-2)$ ☜ Alternatively, just apply the formula on L I 3 I b.

x	$\cdots$	0	$\cdots$	1	$\cdots$	2	$\cdots$
$F'(x)$	$-$	0	$+$	0	$-$	0	$+$
$F(x)$	↘	0	↗	$\frac{1}{4}$	↘	0	↗

Therefore:

The relative maximum value is $\frac{1}{4}$, at $x=1$.

The relative minimum value is 0, at $x=0$, 2.

L 133a KUMON

Definite Integrals II

1. Find the maximum value and the minimum value of

$F(x) = \int_0^x (t-1)(t-3)dt$, when $0 \leqslant x \leqslant 4$.

[Sol] $F(x) = \int_0^x (t^2-4t+3)dt$

$= \left[\frac{1}{3}t^3 - 2t^2 + 3t\right]_0^x$

$= \frac{1}{3}x^3 - 2x^2 + 3x$

Thus, $F'(x) = (x-1)(x-3)$ ☞ Alternatively, apply the formula on L 131 b.

x	0	$\cdots$	1	$\cdots$	3	$\cdots$	4
$F'(x)$	$+$	$+$	0	$-$	0	$+$	$+$
$F(x)$	0	↗	$\frac{4}{3}$	↘	0	↗	$\frac{4}{3}$

Therefore:

The maximum value is $\frac{4}{3}$, at $x = 1,\ 4$.

The minimum value is 0, at $x = 0,\ 3$.

2. Find the maximum value and the minimum value of

$F(x) = -\int_0^x (2t+3)(2t-1)dt$, when $0 \leqslant x \leqslant 3$.

[Sol] $F(x) = -\int_0^x (4t^2+4t-3)dt$

$$= -\left[\frac{4}{3}t^3+2t^2-3t\right]_0^x$$

$$= -\frac{4}{3}x^3-2x^2+3x$$

Thus, $F'(x) = -(2x+3)(2x-1)$ ☜ Alternatively, apply the formula on L131b.

x	0	$\cdots$	$\frac{1}{2}$	$\cdots$	3
$F'(x)$	$+$	$+$	0	$-$	$-$
$F(x)$	0	↗	$\frac{5}{6}$	↘	-45

Therefore:

The maximum value is $\frac{5}{6}$, at $x = \frac{1}{2}$.

The minimum value is -45, at $x = 3$.

L134a KUMON

Definite Integrals II

1. Find the maximum value and the minimum value of

$F(x)=\int_0^x t(t-1)(t-2)dt$, when $0 \leqq x \leqq 3$.

Then, draw the graph of $y=F(x)$.

[Sol] $F(x)=\int_0^x (t^3-3t^2+2t)dt$

$$=\left[\frac{1}{4}t^4-t^3+t^2\right]_0^x$$

$$=\frac{1}{4}x^4-x^3+x^2$$

Thus, $F'(x)=x(x-1)(x-2)$

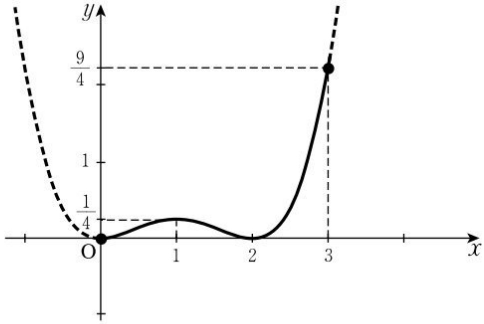
Alternatively, apply the formula on L131b.

x	0	$\cdots$	1	$\cdots$	2	$\cdots$	3
$F'(x)$	0	$+$	0	$-$	0	$+$	$+$
$F(x)$	0	$\nearrow$	$\frac{1}{4}$	$\searrow$	0	$\nearrow$	$\frac{9}{4}$

Therefore:

The maximum value is $\frac{9}{4}$, at $x=3$.

The minimum value is 0, at $x=0,\ 2$.



L 134b

2. Let $f(x)=4x^3+3x^2+x$. Find the value of x for which $F(x)=\int_x^2 f(t)dt$ is the maximum, and the maximum value of $F(x)$.

[Sol] $F(x)=\int_x^2(4t^3+3t^2+t)dt$

$$=\left[t^4+t^3+\frac{1}{2}t^2\right]_x^2$$

$$=(16+8+2)-\left(x^4+x^3+\frac{1}{2}x^2\right)$$

$$=-x^4-x^3-\frac{1}{2}x^2+26$$

Thus, $F'(x)=-4x^3-3x^2-x$

$$=-x(4x^2+3x+1)$$

As $4x^2+3x+1=4\left(x+\frac{3}{8}\right)^2+\frac{7}{16}>0$,

$F'(x)$ changes sign $(+\,,\ -)$ before and after $x=0$.

x	$\cdots$	0	$\cdots$
$F'(x)$	$+$	0	$-$
$F(x)$	↗	26	↘

Therefore, **$F(x)$ has a maximum value of 26, at $x=0$.**

L 135a KUMON

Definite Integrals II

Find the function $f(x)$ and the value(s) of a that satisfies the following equation.

Ex.

$$\int_a^x f(t)dt = 3x^2 - 2x - a \quad \cdots ①$$

[Sol] Differentiating both sides with respect to x,

LHS: $\dfrac{d}{dx}\displaystyle\int_a^x f(t)dt = f(x)$

$$f(x) = 6x - 2$$

Thus,

$$\int_a^x f(t)dt = \int_a^x (6t-2)dt$$

Replacing x by t in $f(x)$ gives $f(t) = 6t - 2$.

$$= \Big[3t^2 - 2t\Big]_a^x$$

$$= 3x^2 - 2x - (3a^2 - 2a) \quad \cdots ②$$

Since ① and ② are equal,

$$3x^2 - 2x - \boxed{\boldsymbol{a}} = 3x^2 - 2x - (\ \boldsymbol{3a^2 - 2a}\ )$$

Therefore,

$$a = 3a^2 - 2a$$

$$3a(a-1) = 0$$

Thus,

$$a = \boxed{\mathbf{0}}\ ,\ \boxed{\mathbf{1}}$$

Answers: a, $3a^2 - 2a$ (in order) 0, 1 (in any order)

L135b

(1) $\displaystyle\int_a^x f(t)dt = x^2-2x-3 \quad \cdots ①$

[Sol] Differentiating both sides with respect to x,

$$\boldsymbol{f(x)=2x-2}$$

Thus,

$$\int_a^x f(t)dt = \int_a^x (2t-2)dt$$

$$= \Big[t^2-2t\Big]_a^x$$

$$= x^2-2x-(a^2-2a) \quad \cdots ②$$

Since ① and ② are equal,

$$x^2-2x-3 = x^2-2x-(a^2-2a)$$

Therefore,

$$3=a^2-2a$$

$$(a-3)(a+1)=0$$

Thus,

$$\boldsymbol{a=3,\ -1}$$

Let's try this!

Referring to the example on side a, another way to find a is as follows:

Substituting $x=a$ into both sides of $\displaystyle\int_a^x f(t)dt = 3x^2-2x-a$,

$$\int_a^a f(t)dt = 3a^2-2a-a$$

$$\boxed{\ \mathbf{0}\ } = 3a^2-3a = 3a(a-1)$$

Therefore, $a=0,\ 1$

Answer: 0

L 136a KUMON

Definite Integrals II

Find the function $f(x)$ that satisfies the following condition.

Ex.

$$f(x)=x-\frac{1}{2}\int_0^1 f(x)dx$$

[Sol] Let $a=\int_0^1 f(x)dx \cdots ①$ $\int_0^1 f(x)dx$ is a constant.

$$f(x)=x-\frac{a}{2} \quad \cdots ②$$

Substituting ② into ①,

$$a=\int_0^1\left(x-\boxed{\frac{a}{2}}\right)dx$$

$$=\left[\frac{1}{2}x^2-\boxed{\frac{a}{2}}x\right]_0^1$$

$$=\frac{1}{2}-\boxed{\frac{a}{2}}$$

Therefore,

$$\frac{3}{2}a=\frac{1}{2}$$

$$a=\frac{1}{3}$$

Thus, $f(x)=\boxed{x-\frac{1}{6}}$

Answers: in order $\frac{a}{2}$, $\frac{a}{2}$, $\frac{a}{2}$, $x-\frac{1}{6}$

L136b

(1) $f(x)=x^2+\int_0^1 xf(t)dt$

Hint

[Sol] $f(x)=x^2+x\int_0^1 f(t)dt$

Let $a=\int_0^1 f(t)dt \quad \cdots ①$

$$f(x)=x^2+ax$$

Replacing x by t in $f(x)$ gives

$$f(t)=t^2+at \quad \cdots ②$$

Substituting ② into ①,

$$a=\int_0^1 (t^2+at)dt$$

$$=\left[\frac{1}{3}t^3+\frac{a}{2}t^2\right]_0^1$$

$$=\frac{1}{3}+\frac{a}{2}$$

Therefore,

$$\frac{1}{2}a=\frac{1}{3}$$

$$a=\frac{2}{3}$$

Thus, $\boldsymbol{f(x)=x^2+\frac{2}{3}x}$

Hint

Here, x is a constant with respect to the variable of integration (t), so

$$\int_0^1 xf(t)dt=x\int_0^1 f(t)dt$$

L 137a KUMON

Definite Integrals II

1. Find the quadratic function $f(x)$ that satisfies the conditions:
$f(-1)=2$, $f'(0)=0$ and $\int_0^1 f(x)dx=-2$.

[Sol] Let $f(x)=ax^2+bx+c$ (where $a\neq 0$)

$$f'(x)=\mathbf{2ax+b}$$

From the given conditions,

$$f(-1)=a-b+c=\mathbf{2}\cdots①$$

$$f'(0)=b=\mathbf{0}\qquad\cdots②$$

$$\int_0^1(ax^2+bx+c)dx=\left[\frac{a}{3}x^3+\frac{b}{2}x^2+cx\right]_0^1$$

$$=\frac{a}{3}+\frac{b}{2}+c=\mathbf{-2}$$

Thus, $2a+3b+6c=-12\quad\cdots③$

From ①, ② and ③,

$$a=6,\ b=0,\ c=-4$$

Therefore, $\boldsymbol{f(x)=6x^2-4}$

L137b

2. Find the quadratic function $f(x)$ that satisfies the conditions:

   $f'(0)=-6$, $f(1)=1$ and $\int_0^2 xf(x)dx=6$.

[Sol] Let $f(x)=ax^2+bx+c$ (where $a\neq 0$)

$$f'(x)=2ax+b$$

From the given conditions,

$$f'(0)=b=-6 \quad \cdots ①$$

$$f(1)=a+b+c=1 \quad \cdots ②$$

$$\int_0^2 (ax^3+bx^2+cx)dx=\left[\frac{a}{4}x^4+\frac{b}{3}x^3+\frac{c}{2}x^2\right]_0^2$$

$$=4a+\frac{8}{3}b+2c=6$$

Thus, $12a+8b+6c=18 \quad \cdots ③$

From ①, ② and ③,

$$a=4,\ b=-6,\ c=3$$

Therefore, $\boldsymbol{f(x)=4x^2-6x+3}$

L 138a KUMON

Definite Integrals II

Ex.

Let $f(x)=x^2+ax+b$. Given any linear function $g(x)$, find the values of a and b for which $\int_{-1}^{1} f(x)g(x)dx=0$ is always true.

[Sol] Let $g(x)=px+q$ (where p and q are constants, and $p\neq 0$)

$$\int_{-1}^{1} f(x)g(x)dx$$

$$=\int_{-1}^{1}(x^2+ax+b)(px+q)dx$$

$$=p\int_{-1}^{1}(x^3+ax^2+bx)dx+q\int_{-1}^{1}(x^2+ax+b)dx$$

$$=2p\int_{0}^{1}ax^2dx+2q\int_{0}^{1}(\ \boldsymbol{x^2+b}\ )dx$$

$$=2p\left[\frac{a}{3}x^3\right]_0^1+2q\left[\ \boldsymbol{\frac{1}{3}x^3+bx}\ \right]_0^1$$

$$=\frac{2a}{3}p+\frac{2}{3}(\ \boldsymbol{1+3b}\ )q=0 \quad \cdots①$$

In order for ① to be true, regardless of the values of p and q,

$$\frac{2a}{3}=0,\ \ \frac{2}{3}(\ \boldsymbol{1+3b}\ )=0$$

Therefore,

$$a=0,\ \ b=\boxed{\boldsymbol{-\frac{1}{3}}}$$

Answers: in order x^2+b, $\frac{1}{3}x^3+bx$, $1+3b$, $1+3b$, $-\frac{1}{3}$

L138b

1. Let $f(x)=ax^2+bx+1$. Given any linear function $g(x)$, find the values of a and b for which $\int_0^1 f(x)g(x)dx=0$ is always true.

[Sol] Let $g(x)=px+q$ (where p and q are constants, and $p\neq 0$)

$$\begin{aligned}
&\int_0^1 f(x)g(x)dx\\
&=\int_0^1 (ax^2+bx+1)(px+q)dx\\
&=p\int_0^1 (ax^3+bx^2+x)dx+q\int_0^1 (ax^2+bx+1)dx\\
&=p\left[\frac{a}{4}x^4+\frac{b}{3}x^3+\frac{1}{2}x^2\right]_0^1+q\left[\frac{a}{3}x^3+\frac{b}{2}x^2+x\right]_0^1\\
&=\left(\frac{a}{4}+\frac{b}{3}+\frac{1}{2}\right)p+\left(\frac{a}{3}+\frac{b}{2}+1\right)q=0 \quad \cdots①
\end{aligned}$$

In order for ① to be true, regardless of the values of p and q,

$$\frac{a}{4}+\frac{b}{3}+\frac{1}{2}=0 \quad \cdots②$$

$$\frac{a}{3}+\frac{b}{2}+1=0 \quad \cdots③$$

From ② and ③,

$$\boldsymbol{a=6,\ \ b=-6}$$

L 139a KUMON

Definite Integrals II

Ex.

Given that $f(x)$ is a linear function, and $\int_0^1 f(x)dx = 1$, prove that $\int_0^1 [f(x)]^2 dx > 1$.

[Sol] Let $f(x) = ax + b$ (where $a \neq 0$)

$$\int_0^1 f(x)dx = \int_0^1 (ax+b)dx = \left[\frac{a}{2}x^2 + bx\right]_0^1$$

$$= \frac{a}{2} + b = \boxed{1} \quad \cdots ①$$

$$\int_0^1 [f(x)]^2 dx = \int_0^1 (ax+b)^2 dx = \int_0^1 (a^2x^2 + 2abx + b^2)dx$$

$$= \left[\frac{a^2}{3}x^3 + abx^2 + b^2x\right]_0^1 = \frac{a^2}{3} + ab + b^2 \quad \cdots ②$$

From ①, $b = \boxed{\mathbf{1-\frac{a}{2}}} \quad \cdots ③$

Substituting ③ into ②,

$$\int_0^1 [f(x)]^2 dx = \frac{a^2}{3} + a\left(\boxed{\mathbf{1-\frac{a}{2}}}\right) + \left(\boxed{\mathbf{1-\frac{a}{2}}}\right)^2$$

$$= \frac{1}{12}a^2 + \boxed{1} > 1$$

Therefore, $\int_0^1 [f(x)]^2 dx > 1$

Answers: in order 1, $1-\frac{a}{2}$, $1-\frac{a}{2}$, $1-\frac{a}{2}$, 1

L 139b

1. Given that $f(x)$ is a linear function, prove that $\left[\int_0^1 f(x)dx\right]^2 < \int_0^1 [f(x)]^2 dx$.

[Sol] Let $f(x) = ax + b$ (where $a \neq 0$)

$$\int_0^1 f(x)dx = \int_0^1 (ax+b)dx$$

$$= \left[\frac{a}{2}x^2 + bx\right]_0^1$$

$$= \frac{a}{2} + b$$

$$\left[\int_0^1 f(x)dx\right]^2 = \left(\frac{a}{2} + b\right)^2 = \frac{a^2}{4} + ab + b^2$$

$$\int_0^1 [f(x)]^2 dx = \int_0^1 (a^2x^2 + 2abx + b^2)dx$$

$$= \left[\frac{a^2}{3}x^3 + abx^2 + b^2x\right]_0^1$$

$$= \frac{a^2}{3} + ab + b^2$$

$$\int_0^1 [f(x)]^2 dx - \left[\int_0^1 f(x)dx\right]^2 = \frac{a^2}{3} + ab + b^2 - \left(\frac{a^2}{4} + ab + b^2\right)$$

$$= \frac{1}{12}a^2 > 0$$

Therefore, $\left[\int_0^1 f(x)dx\right]^2 < \int_0^1 [f(x)]^2 dx$

L 140a KUMON

Definite Integrals II

1. Find the relative extreme values of $f(x)=\int_0^x 2(t+1)(t-3)dt$.

[Sol] $f(x)=\left[\frac{2}{3}t^3-2t^2-6t\right]_0^x$

$$=\frac{2}{3}x^3-2x^2-6x$$

Thus, $f'(x)=2(x+1)(x-3)$

x	$\cdots$	-1	$\cdots$	3	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	$\frac{10}{3}$	$\searrow$	-18	$\nearrow$

The relative maximum value is $\frac{10}{3}$, at $x=-1$.

The relative minimum value is -18, at $x=3$.

L 140b

2. Find the function $f(x)$ and the value(s) of a that satisfies the following equation.

$$\int_a^x f(t)dt = 2x^2 - 3x + 1 \quad \cdots ①$$

[Sol] Differentiating both sides with respect to x,

$$\boldsymbol{f(x) = 4x - 3}$$

Thus,

$$\int_a^x f(t)dt = \int_a^x (4t-3)dt$$
$$= \Big[2t^2 - 3t\Big]_a^x$$
$$= 2x^2 - 3x - (2a^2 - 3a) \quad \cdots ②$$

Since ① and ② are equal,

$$2x^2 - 3x + 1 = 2x^2 - 3x - (2a^2 - 3a)$$

Therefore,

$$1 = -2a^2 + 3a$$
$$2a^2 - 3a + 1 = 0$$
$$(2a-1)(a-1) = 0$$

Thus,

$$\boldsymbol{a = \frac{1}{2},\ 1}$$

Alternative Solution

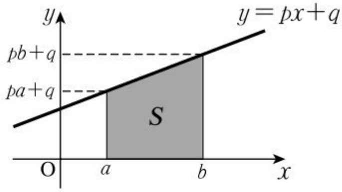
Substituting $x = a$ into ①,

$$\int_a^a f(t)dt = 2a^2 - 3a + 1$$
$$0 = 2a^2 - 3a + 1$$
$$(2a-1)(a-1) = 0$$
$$a = \frac{1}{2},\ 1$$

L 141a KUMON

Areas I

Consider the area enclosed by the linear function $f(x)=px+q$ (where $p\neq 0$), the x-axis, and the lines $x=a$ and $x=b$.



(i) Calculating the area directly,

$S=$ (distance between parallel sides) $\times$ (average of parallel sides)

$$=(b-a)\cdot\frac{1}{2}[(pa+q)+(pb+q)]$$

$$=(b-a)\cdot\left[\frac{p(a+b)}{2}+q\right]$$

$$=\frac{p}{2}(b^2-a^2)+q(b-a)$$

Use the formula for the area of a trapezium.

(ii) Considering the following integral,

$$\int_a^b f(x)\,dx=\int_a^b (px+q)dx$$

$$=\left[\frac{p}{2}x^2+qx\right]_a^b=\frac{p}{2}(b^2-a^2)+q(b-a)$$

Comparing the results of (i) and (ii), $S=\int_a^b f(x)\,dx$

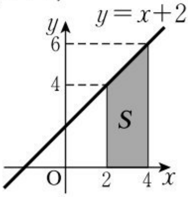
$$\left(\begin{array}{l}\text{the area of the}\\ \text{enclosed region}\end{array}\right)=\left(\begin{array}{l}\text{the definite integral in}\\ \text{the specified region}\end{array}\right)$$

1. Find the area, S, enclosed by the line $f(x)=x+2$ and the x-axis in the interval $[2,\ 4]$, using two different methods:

(1) Using the formula for the area of a trapezium,

[Sol] $\boldsymbol{S=2\times\frac{1}{2}(4+6)}$

$\boldsymbol{=10}$



(2) Using integration,

[Sol]

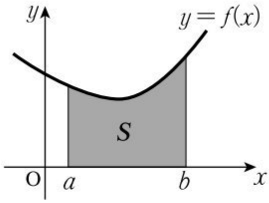
$$\int_2^4 \boldsymbol{(x+2)dx}=\left[\frac{1}{2}x^2+2x\right]_2^4$$

$$=\left(\frac{16}{2}+8\right)-\left(\frac{4}{2}+4\right)$$

$$=16-6\ \boldsymbol{=10}$$

L 141b

Generally when $f(x) \geqslant 0$ we can find the area, S, enclosed by the curve $y = f(x)$ and the x-axis in the interval $[a,\ b]$ using the following formula:



$$\boxed{S = \int_a^b f(x)dx}$$

2. Find the area, S, enclosed by the given curve and the x-axis as follows:

Ex.

$y = -x^2 - 3x$

[Sol] From $y = -x^2 - 3x = -x(x+3) = 0$, ☜ Find the x-intercepts.

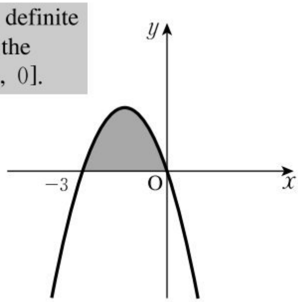
$x = -3,\ 0$

Therefore,

$$S = \int_{-3}^{0} (-x^2 - 3x)dx$$ ☜ Calculate the definite integral over the interval $[-3,\ 0]$.

$$= -\left[\frac{1}{3}x^3 + \frac{3}{2}x^2\right]_{-3}^{0}$$

$$= -9 + \frac{27}{2} = \frac{9}{2}$$



(1) $y = -x^2 + 2x + 3$

[Sol] From $y = -x^2 + 2x + 3 = -(x-3)(x+1) = 0$,

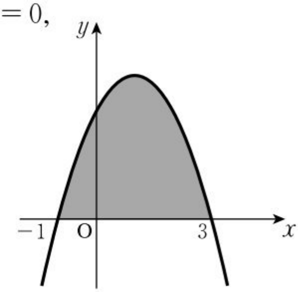
$x = -1,\ 3$

Therefore,

$$S = \int_{-1}^{3} (-x^2 + 2x + 3)dx$$

$$= \left[-\frac{1}{3}x^3 + x^2 + 3x\right]_{-1}^{3}$$

$$= (-9 + 9 + 9) - \left(\frac{1}{3} + 1 - 3\right) = \mathbf{\frac{32}{3}}$$

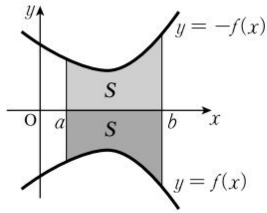


L 142a KUMON

Areas I

In the interval $[a,\ b]$ there are two regions, one enclosed by $y=f(x)$ and the x-axis and one enclosed by $y=-f(x)$ and the x-axis, which are symmetric with respect to the x-axis and equal in area.
Therefore, when $f(x)\leqslant 0$ we can find the area using the following formula:

$$\boxed{S=-\int_a^b f(x)dx}$$



1. Find the area, S, enclosed by the given curve and the x-axis as follows:

Ex.

$y=x^2-4x$

[Sol] From $y=x^2-4x=x(x-4)=0$,

$x=0,\ 4$

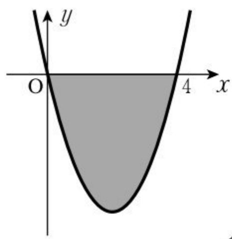
Therefore,

$$S=-\int_0^4(x^2-4x)dx$$

Since the region is below the x-axis in the interval $[0,\ 4]$, put a negative sign, "$-$", in front of the integral.

$$=-\left[\frac{1}{3}x^3-2x^2\right]_0^4$$

$$=-\left(\frac{64}{3}-32\right)=\frac{32}{3}$$



(1) $y=x^2+x-2$

[Sol] From $y=x^2+x-2=(x+2)(x-1)=0$,

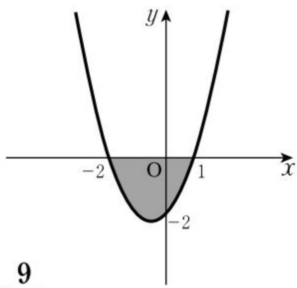
$x=-2,\ 1$

Therefore,

$$S=-\int_{-2}^1(x^2+x-2)dx$$

$$=-\left[\frac{1}{3}x^3+\frac{1}{2}x^2-2x\right]_{-2}^1$$

$$=-\left[\left(\frac{1}{3}+\frac{1}{2}-2\right)-\left(-\frac{8}{3}+2+4\right)\right]=\boldsymbol{\frac{9}{2}}$$



L 142b

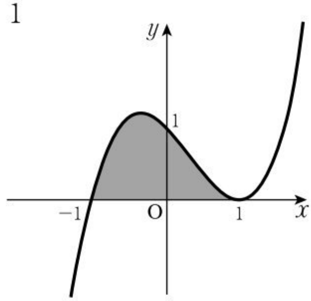
2. Find the area, S, enclosed by the given curve and the x-axis.

(1) $y=(x+1)(x-1)^2$

[Sol] From $y=(x+1)(x-1)^2=0$, $x=-1, 1$

Therefore,

$$\boldsymbol{S}=\int_{-1}^{1}(x+1)(x-1)^2dx$$
$$=\int_{-1}^{1}(x^3-x^2-x+1)dx$$
$$=2\int_{0}^{1}(-x^2+1)dx$$
$$=2\left[-\frac{1}{3}x^3+x\right]_0^1$$
$$=2\left(-\frac{1}{3}+1\right)=\boldsymbol{\frac{4}{3}}$$

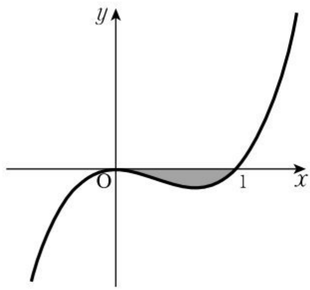


(2) $y=x^2(x-1)$

[Sol] From $y=x^2(x-1)=0$, $x=0, 1$

Therefore,

$$\boldsymbol{S}=-\int_{0}^{1}x^2(x-1)dx$$
$$=-\int_{0}^{1}(x^3-x^2)dx$$
$$=-\left[\frac{1}{4}x^4-\frac{1}{3}x^3\right]_0^1$$
$$=-\left(\frac{1}{4}-\frac{1}{3}\right)=\boldsymbol{\frac{1}{12}}$$



Note: Given that the curve $y=f(x)$ intersects the x-axis at 2 points, where the x-coordinates are α and β (where $\alpha<\beta$), the area, S, enclosed by the curve and the x-axis can be expressed as follows:

(i) When the region is above the x-axis: $\boldsymbol{S}=\int_{\alpha}^{\beta}f(x)dx$

(ii) When the region is below the x-axis: $\boldsymbol{S}=-\int_{\alpha}^{\beta}f(x)dx$

L 143a KUMON

Areas I

1. Find the area, S, enclosed by the given curve and the x-axis as follows:

Ex.

$y=x^3-3x^2+2x$

[Sol]

From $y=x^3-3x^2+2x=x(x-2)(x-1)=0$,

$x=0,\ 1,\ 2$

Therefore,

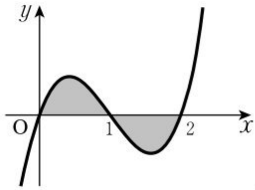
$$S=\int_0^1(x^3-3x^2+2x)dx-\int_1^2(x^3-3x^2+2x)dx$$

Since $y\leqslant 0$ in the interval $[1,\ 2]$, put a negative sign, "$-$", in front of this integral.

$$=\left[\frac{1}{4}x^4-x^3+x^2\right]_0^1-\left[\frac{1}{4}x^4-x^3+x^2\right]_1^2$$

$$=\left(\frac{1}{4}-1+1\right)-\left[(4-8+4)-\left(\frac{1}{4}-1+1\right)\right]$$

$$=\frac{1}{2}$$



(1) $y=-x^3-x^2+2x$

[Sol] From $y=-x^3-x^2+2x=-x(x+2)(x-1)=0$,

$x=-2,\ 0,\ 1$

Therefore,

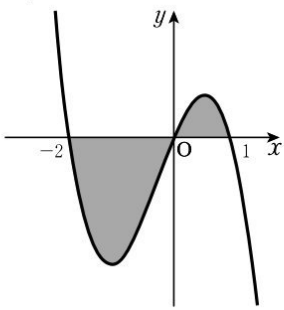
$$S=-\int_{-2}^0(-x^3-x^2+2x)dx+\int_0^1(-x^3-x^2+2x)dx$$

$$=-\left[-\frac{1}{4}x^4-\frac{1}{3}x^3+x^2\right]_{-2}^0$$

$$+\left[-\frac{1}{4}x^4-\frac{1}{3}x^3+x^2\right]_0^1$$

$$=\left(-4+\frac{8}{3}+4\right)+\left(-\frac{1}{4}-\frac{1}{3}+1\right)$$

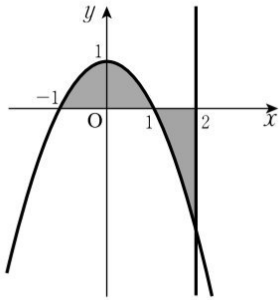
$$=\mathbf{\frac{37}{12}}$$



L 143b

2. Find the area, *S*, of the shaded region (▭) enclosed by the given curve, the given line and the x-axis.

(1) $y=-x^2+1$, line $x=2$



[Sol] From $y=-x^2+1=-(x+1)(x-1)=0$,

$x=-1,\ 1$

Therefore,

$$\boldsymbol{S}=\int_{-1}^{1}(-x^2+1)dx-\int_{1}^{2}(-x^2+1)dx$$

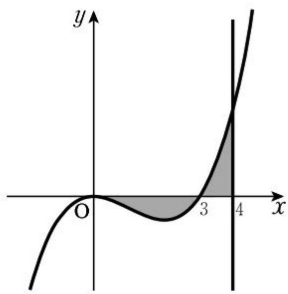
$$=2\int_{0}^{1}(-x^2+1)dx-\int_{1}^{2}(-x^2+1)dx$$

$$=2\left[-\frac{1}{3}x^3+x\right]_0^1-\left[-\frac{1}{3}x^3+x\right]_1^2$$

$$=2\left(-\frac{1}{3}+1\right)-\left[\left(-\frac{8}{3}+2\right)-\left(-\frac{1}{3}+1\right)\right]$$

$$=\boldsymbol{\frac{8}{3}}$$

(2) $y=\frac{1}{2}x^2(x-3)$, line $x=4$



[Sol] From $y=\frac{1}{2}x^2(x-3)=0$, $x=0,\ 3$

Therefore,

$$\boldsymbol{S}=-\int_{0}^{3}\frac{1}{2}x^2(x-3)dx+\int_{3}^{4}\frac{1}{2}x^2(x-3)dx$$

$$=-\int_{0}^{3}\left(\frac{1}{2}x^3-\frac{3}{2}x^2\right)dx+\int_{3}^{4}\left(\frac{1}{2}x^3-\frac{3}{2}x^2\right)dx$$

$$=-\left[\frac{1}{8}x^4-\frac{1}{2}x^3\right]_0^3+\left[\frac{1}{8}x^4-\frac{1}{2}x^3\right]_3^4$$

$$=-\left(\frac{81}{8}-\frac{27}{2}\right)+\left(\frac{256}{8}-\frac{64}{2}\right)-\left(\frac{81}{8}-\frac{27}{2}\right)$$

$$=\boldsymbol{\frac{27}{4}}$$

L 144a KUMON

Areas I

1. Find the area, S, enclosed by the curve $y=x^2+4x$ and the x-axis in the interval $-1 \leqslant x \leqslant 1$.

[Sol] From $y=x^2+4x=x(x+4)=0$, $x=-4, 0$

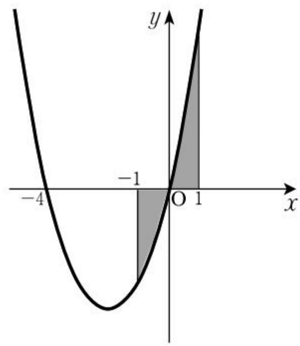
Therefore,

$$\boldsymbol{S}=-\int_{-1}^{0}(x^2+4x)dx+\int_{0}^{1}(x^2+4x)dx$$

$$=-\left[\frac{1}{3}x^3+2x^2\right]_{-1}^{0}+\left[\frac{1}{3}x^3+2x^2\right]_{0}^{1}$$

$$=-\left[-\left(-\frac{1}{3}+2\right)\right]+\left(\frac{1}{3}+2\right)$$

$$\boldsymbol{=4}$$



2. Find the area, S, enclosed by the curve $y=-x^2+6x-5$ and the x-axis in the interval $0 \leqslant x \leqslant 4$.

[Sol] From $y=-x^2+6x-5=-(x-5)(x-1)=0$, $x=1, 5$

Therefore,

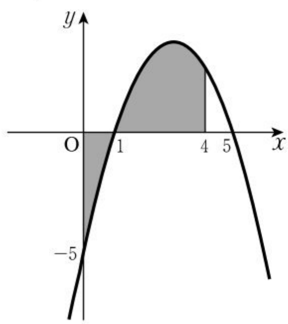
$$\boldsymbol{S}=-\int_{0}^{1}(-x^2+6x-5)dx+\int_{1}^{4}(-x^2+6x-5)dx$$

$$=-\left[-\frac{1}{3}x^3+3x^2-5x\right]_{0}^{1}+\left[-\frac{1}{3}x^3+3x^2-5x\right]_{1}^{4}$$

$$=-\left(-\frac{1}{3}+3-5\right)+\left(-\frac{64}{3}+48-20\right)$$

$$-\left(-\frac{1}{3}+3-5\right)$$

$$\boldsymbol{=\frac{34}{3}}$$

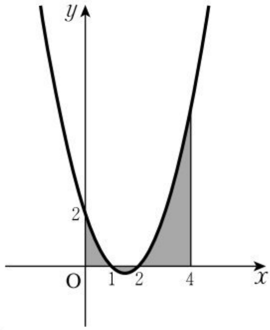


L 144b

3. Find the area, S, enclosed by the curve $y=x^2-3x+2$ and the x-axis in the interval $0 \leqslant x \leqslant 4$.

[Sol] From $y=x^2-3x+2=(x-2)(x-1)=0$,

$x=1, \ 2$



Therefore,

$$\boldsymbol{S}=\int_0^1(x^2-3x+2)dx-\int_1^2(x^2-3x+2)dx+\int_2^4(x^2-3x+2)dx$$

$$=\left[\frac{1}{3}x^3-\frac{3}{2}x^2+2x\right]_0^1-\left[\frac{1}{3}x^3-\frac{3}{2}x^2+2x\right]_1^2+\left[\frac{1}{3}x^3-\frac{3}{2}x^2+2x\right]_2^4$$

$$=\left(\frac{1}{3}-\frac{3}{2}+2\right)-\left[\left(\frac{8}{3}-6+4\right)-\left(\frac{1}{3}-\frac{3}{2}+2\right)\right]$$

$$+\left(\frac{64}{3}-24+8\right)-\left(\frac{8}{3}-6+4\right)$$

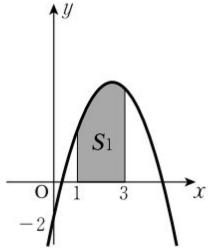
$$=\mathbf{\frac{17}{3}}$$

L 145a KUMON

Areas I

1. Complete the following questions.

(1) Find the area, S_1, enclosed by $y=-x^2+5x-2$, the x-axis, $x=1$ and $x=3$.

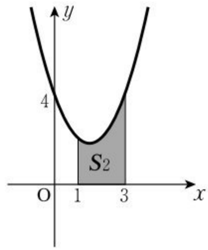


[Sol] $S_1=\int_1^3(-x^2+5x-2)dx$

$$=\left[-\frac{1}{3}x^3+\frac{5}{2}x^2-2x\right]_1^3$$

$$=\left(-9+\frac{45}{2}-6\right)-\left(-\frac{1}{3}+\frac{5}{2}-2\right)=\mathbf{\frac{22}{3}}$$

(2) Find the area, S_2, enclosed by $y=x^2-3x+4$, the x-axis, $x=1$ and $x=3$.



[Sol] $S_2=\int_1^3(x^2-3x+4)dx$

$$=\left[\frac{1}{3}x^3-\frac{3}{2}x^2+4x\right]_1^3$$

$$=\left(9-\frac{27}{2}+12\right)-\left(\frac{1}{3}-\frac{3}{2}+4\right)=\mathbf{\frac{14}{3}}$$

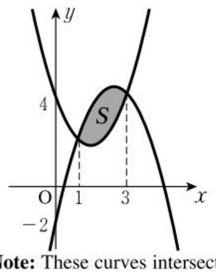
(3) Find the area, S, enclosed by $y=-x^2+5x-2$ and $y=x^2-3x+4$.

[Sol] The area, S, is equal to the area S_1-S_2.

Therefore,

$$S=S_1-S_2=\frac{22}{3}-\frac{14}{3}=\mathbf{\frac{8}{3}}$$

Use the answers from (1) and (2).



[**Note:** These curves intersect where $x=1$ and $x=3$.]

2. Fill in the blank box.

Generally when $f(x)\geqq g(x)$, the area, S, enclosed by the two curves $y=f(x)$ and $y=g(x)$ in the interval $[a,\ b]$ can be found using the following formula:

$$S=\int_a^b f(x)dx-\int_a^b \boxed{\boldsymbol{g(x)}}\,dx=\int_a^b[f(x)-g(x)]dx$$

L 145b

3. Find the area, S, enclosed by the given curve and the given line.

Ex.

$y=x^2, \quad y=x+2$

[Sol] Finding the x-coordinates of the points of intersection,

from $x^2=x+2$,

$(x-2)(x+1)=0$

$x=-1,\ 2$ ☜ The x-coordinates of the points of intersection become the interval range.

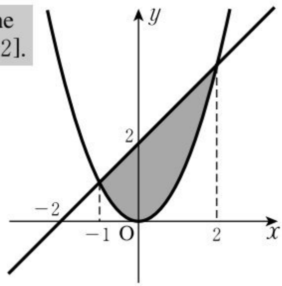
Therefore,

$S=\int_{-1}^{2}(x+2-x^2)dx$ ☜ $x+2 \geqq x^2$ in the interval $[-1,\ 2]$.

$=\left[\frac{1}{2}x^2+2x-\frac{1}{3}x^3\right]_{-1}^{2}$

$=\left(2+4-\frac{8}{3}\right)-\left(\frac{1}{2}-2+\frac{1}{3}\right)$

$=\frac{9}{2}$



(1) $y=-x^2+2x+3, \quad y=x+1$

[Sol] From $-x^2+2x+3=x+1$,

$(x-2)(x+1)=0$

$x=-1,\ 2$

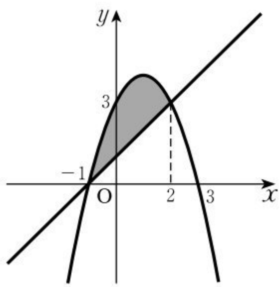
Therefore,

$S=\int_{-1}^{2}[-x^2+2x+3-(x+1)]dx$

$=\int_{-1}^{2}(-x^2+x+2)dx$

$=\left[-\frac{1}{3}x^3+\frac{1}{2}x^2+2x\right]_{-1}^{2}$

$=\left(-\frac{8}{3}+2+4\right)-\left(\frac{1}{3}+\frac{1}{2}-2\right)=\mathbf{\frac{9}{2}}$



L 146a KUMON

Areas I

As shown in Figure (i), in the interval $[a,\ b]$, $f(x) \geqq g(x)$ but $g(x)$ has negative values. As shown in Figure (ii), if we translate $f(x)$ and $g(x)$, C units in the direction of the positive y-axis, the area, S, of the region is unchanged. Thus, the area is:

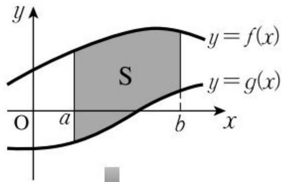
$$S = \int_a^b [(f(x)+C)-(g(x)+C)]\,dx$$

$$= \int_a^b [f(x)-g(x)]\,dx$$

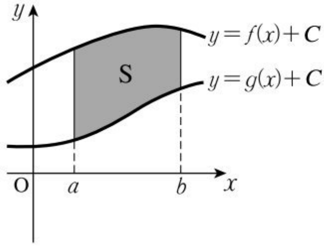
Therefore, the area, enclosed by $y=f(x)$ and $y=g(x)$ is:

$$S = \int_a^b [f(x)-g(x)]\,dx \quad (\text{where } f(x) \geqq g(x))$$

[Figure (i)]



[Figure (ii)]



1. Find the area, S, enclosed by the given curve(s) and the given line.

(1) $y = x^2 - x - 1,\qquad y = x + 2$

[Sol] From $x^2 - x - 1 = x + 2$,

$(x-3)(x+1) = 0$

$x = -1,\ 3$

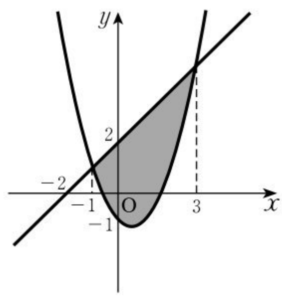
Therefore,

$$S = \int_{-1}^{3} [x+2-(x^2-x-1)]\,dx$$

$$= \int_{-1}^{3} (-x^2+2x+3)\,dx$$

$$= \left[-\frac{1}{3}x^3 + x^2 + 3x\right]_{-1}^{3}$$

$$= (-9+9+9) - \left(\frac{1}{3}+1-3\right) = \mathbf{\frac{32}{3}}$$



L 146b

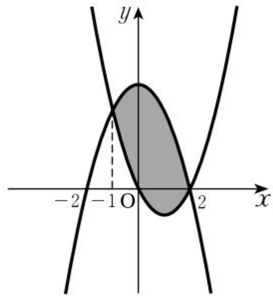
(2) $y = x^2 - 2x, \quad y = -x^2 + 4$

[Sol] From $x^2 - 2x = -x^2 + 4$,

$$2(x-2)(x+1) = 0$$

$$x = -1,\ 2$$

Therefore,



$$\boldsymbol{S} = \int_{-1}^{2} [-x^2 + 4 - (x^2 - 2x)]\,dx$$

$$= \int_{-1}^{2} (-2x^2 + 2x + 4)\,dx$$

$$= \left[-\frac{2}{3}x^3 + x^2 + 4x\right]_{-1}^{2}$$

$$= \left(-\frac{16}{3} + 4 + 8\right) - \left(\frac{2}{3} + 1 - 4\right) \mathbf{= 9}$$

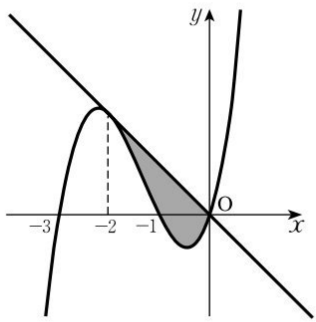
(3) $y = x^3 + 4x^2 + 3x, \quad y = -x$

[Sol] From $x^3 + 4x^2 + 3x = -x$,

$$x^3 + 4x^2 + 4x = x(x+2)^2 = 0$$

$$x = -2,\ 0$$

Therefore,



$$\boldsymbol{S} = \int_{-2}^{0} [-x - (x^3 + 4x^2 + 3x)]\,dx$$

$$= \int_{-2}^{0} (-x^3 - 4x^2 - 4x)\,dx$$

$$= \left[-\frac{1}{4}x^4 - \frac{4}{3}x^3 - 2x^2\right]_{-2}^{0}$$

$$= -\left(-4 + \frac{32}{3} - 8\right)$$

$$\boldsymbol{= \frac{4}{3}}$$

L 147a KUMON

Areas I

Find the area, S, enclosed by the given curve(s) and line(s).

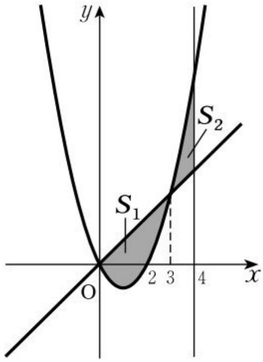
(1) $y=x^2-2x, \quad y=x, \quad x=4$

[Sol] From $x^2-2x=x$,

$x(x-3)=0$

$x=0,\ 3$

Therefore:



(i) When $0 \leqq x \leqq 3$, ☜ $x \geqq x^2-2x$

$$S_1=\int_0^3[x-(x^2-2x)]\,dx$$

$$=\int_0^3(-x^2+3x)\,dx$$

$$=\left[-\frac{1}{3}x^3+\frac{3}{2}x^2\right]_0^3$$

$$=-9+\frac{27}{2}=\mathbf{\frac{9}{2}}$$

(ii) When $3 \leqq x \leqq 4$, ☜ $x^2-2x \geqq x$

$$S_2=\int_3^4[(x^2-2x)-x]\,dx$$

$$=\int_3^4(x^2-3x)\,dx$$

$$=\left[\frac{1}{3}x^3-\frac{3}{2}x^2\right]_3^4$$

$$=\left(\frac{64}{3}-24\right)-\left(9-\frac{27}{2}\right)=\mathbf{\frac{11}{6}}$$

From (i) and (ii),

$$S=S_1+S_2=\frac{9}{2}+\frac{11}{6}=\mathbf{\frac{19}{3}}$$

L 147b

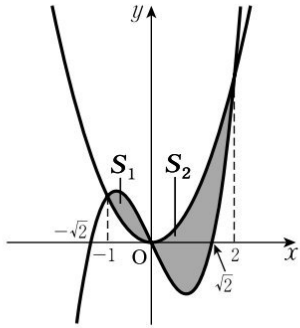
(2) $y=x^3-2x, \quad y=x^2$

[Sol] From $x^3-2x=x^2$,

$$x^3-x^2-2x=x(x+1)(x-2)=0$$

$$x=-1,\ 0,\ 2$$

Therefore:



(i) When $-1 \leqq x \leqq 0$,

$$\begin{aligned} S_1 &= \int_{-1}^{0}[(x^3-2x)-x^2]dx \\ &= \int_{-1}^{0}(x^3-x^2-2x)dx \\ &= \left[\frac{1}{4}x^4-\frac{1}{3}x^3-x^2\right]_{-1}^{0} \\ &= -\left(\frac{1}{4}+\frac{1}{3}-1\right)=\frac{5}{12} \end{aligned}$$

(ii) When $0 \leqq x \leqq 2$,

$$\begin{aligned} S_2 &= \int_{0}^{2}[x^2-(x^3-2x)]dx \\ &= \int_{0}^{2}(-x^3+x^2+2x)dx \\ &= \left[-\frac{1}{4}x^4+\frac{1}{3}x^3+x^2\right]_{0}^{2} \\ &= -4+\frac{8}{3}+4=\frac{8}{3} \end{aligned}$$

From (i) and (ii),

$$S=S_1+S_2=\frac{5}{12}+\frac{8}{3}=\mathbf{\frac{37}{12}}$$

L 148a KUMON

Areas I

The following formula can be used to find the areas of regions involving parabolas.

$$\int_{\alpha}^{\beta}(x-\alpha)(x-\beta)dx=-\frac{1}{6}(\beta-\alpha)^3$$

Find the area, S, enclosed by the given parabolas and/or lines.

Ex.

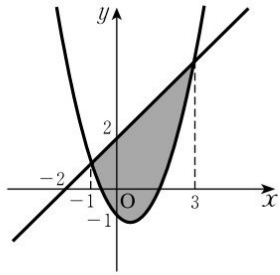
$y=x^2-x-1, \quad y=x+2$

[Sol] From $x^2-x-1=x+2$,

$(x-3)(x+1)=0$

$x=-1,\ 3$

$$\begin{aligned}S&=\int_{-1}^{3}[x+2-(x^2-x-1)]dx\\&=-\int_{-1}^{3}(x^2-2x-3)dx\\&=-\int_{-1}^{3}(x+1)(x-3)dx\\&=\frac{1}{6}[3-(-1)]^3=\frac{32}{3}\end{aligned}$$



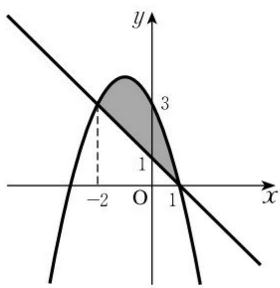
(1) $y=-x^2-2x+3, \quad y=-x+1$

[Sol] From $-x^2-2x+3=-x+1$,

$(x+2)(x-1)=0$

$x=-2,\ 1$

$$\begin{aligned}S&=\int_{-2}^{1}[-x^2-2x+3-(-x+1)]dx\\&=-\int_{-2}^{1}(x^2+x-2)dx\\&=-\int_{-2}^{1}(x+2)(x-1)dx\\&=\frac{1}{6}(1+2)^3=\mathbf{\frac{9}{2}}\end{aligned}$$



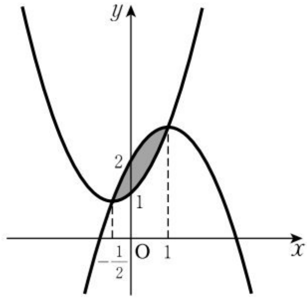
L 148b

(2) $y=x^2+x+1, \quad y=-x^2+2x+2$

[Sol] From $x^2+x+1=-x^2+2x+2$,

$$(2x+1)(x-1)=0$$

$$x=-\frac{1}{2},\ 1$$



$$\boldsymbol{S}=\int_{-\frac{1}{2}}^{1}[-x^2+2x+2-(x^2+x+1)]dx$$

$$=-2\int_{-\frac{1}{2}}^{1}\left(x^2-\frac{1}{2}x-\frac{1}{2}\right)dx$$

$$=-2\int_{-\frac{1}{2}}^{1}\left(x+\frac{1}{2}\right)(x-1)dx$$

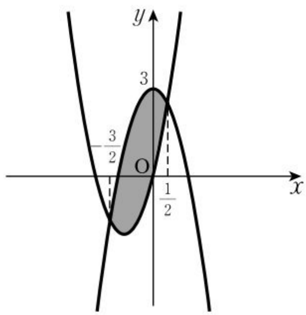
$$=\frac{1}{3}\left(1+\frac{1}{2}\right)^3=\boldsymbol{\frac{9}{8}}$$

(3) $y=2x^2+4x, \quad y=-2x^2+3$

[Sol] From $2x^2+4x=-2x^2+3$,

$$(2x+3)(2x-1)=0$$

$$x=-\frac{3}{2},\ \frac{1}{2}$$



$$\boldsymbol{S}=\int_{-\frac{3}{2}}^{\frac{1}{2}}[-2x^2+3-(2x^2+4x)]dx$$

$$=\int_{-\frac{3}{2}}^{\frac{1}{2}}(-4x^2-4x+3)dx$$

$$=-4\int_{-\frac{3}{2}}^{\frac{1}{2}}\left(x^2+x-\frac{3}{4}\right)dx$$

$$=-4\int_{-\frac{3}{2}}^{\frac{1}{2}}\left(x+\frac{3}{2}\right)\left(x-\frac{1}{2}\right)dx$$

$$=\frac{2}{3}\left(\frac{1}{2}+\frac{3}{2}\right)^3=\boldsymbol{\frac{16}{3}}$$

L 149a KUMON

Areas I

Find the area, S, enclosed by the given parabolas and/or lines.

Ex.

$y=-x^2+3x-1, \quad y=x-2$

[Sol] From $-x^2+3x-1=x-2$,

$x^2-2x-1=0 \quad \cdots ①$

Let α and β (where $\alpha<\beta$) be the solutions of ①, so

$\alpha+\beta=\boxed{2}$, $\alpha\beta=\boxed{-1}$ From the root-coefficient relationships.

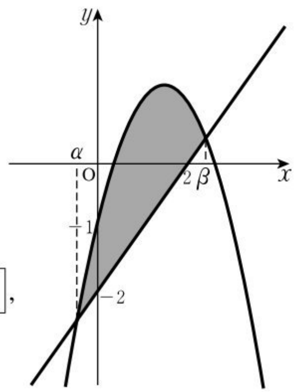
Therefore,

$$S=\int_\alpha^\beta[-x^2+3x-1-(x-2)]dx$$

$$=-\int_\alpha^\beta(x^2-2x-1)dx$$

$$=-\int_\alpha^\beta(x-\alpha)(x-\beta)dx$$

$$=\frac{1}{6}(\beta-\alpha)^3$$



From $(\beta-\alpha)^2=(\alpha+\beta)^2-4\alpha\beta=\boxed{8}$,

$\beta-\alpha=\boxed{2\sqrt{2}}$

Therefore, $(\beta-\alpha)^3=\boxed{16\sqrt{2}}$

Thus, $S=\boxed{\dfrac{8\sqrt{2}}{3}}$

Answers: in order 2, -1, 8, $2\sqrt{2}$, $16\sqrt{2}$, $\frac{8\sqrt{2}}{3}$

L 149b

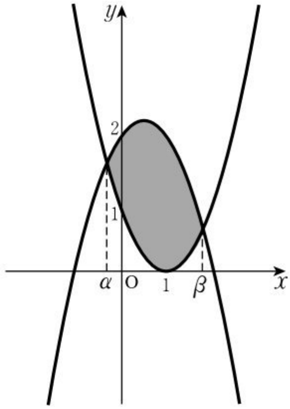
(1) $y=x^2-2x+1,\quad y=-x^2+x+2$

[Sol] From $x^2-2x+1=-x^2+x+2$,

$$2x^2-3x-1=0 \quad \cdots ①$$

Let α and β (where $\alpha<\beta$) be the solutions of ①, so

$$\alpha+\beta=\frac{3}{2},\ \alpha\beta=-\frac{1}{2}$$



Therefore,

$$\begin{aligned}\boldsymbol{S}&=\int_{\alpha}^{\beta}[-x^2+x+2-(x^2-2x+1)]\,dx\\&=-2\int_{\alpha}^{\beta}\left(x^2-\frac{3}{2}x-\frac{1}{2}\right)dx\\&=-2\int_{\alpha}^{\beta}(x-\alpha)(x-\beta)\,dx\\&=\frac{1}{3}(\beta-\alpha)^3\end{aligned}$$

From $(\beta-\alpha)^2=(\alpha+\beta)^2-4\alpha\beta=\dfrac{17}{4}$,

$$\beta-\alpha=\frac{\sqrt{17}}{2}$$

Therefore, $(\beta-\alpha)^3=\dfrac{17\sqrt{17}}{8}$

Thus, $\boldsymbol{S=\dfrac{17\sqrt{17}}{24}}$

L 150a KUMON

Areas I

Find the area, S, enclosed by the given curves and/or lines.

(1) $y=x(x+1)(x-2)$, the x-axis

[Sol] From $y=x(x+1)(x-2)=0$, $x=-1, 0, 2$

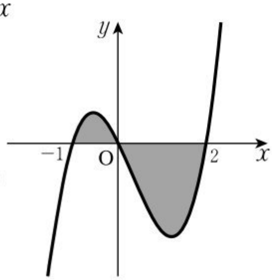
Therefore,

$$S=\int_{-1}^{0}x(x+1)(x-2)dx-\int_{0}^{2}x(x+1)(x-2)dx$$

$$=\int_{-1}^{0}(x^3-x^2-2x)dx-\int_{0}^{2}(x^3-x^2-2x)dx$$

$$=\left[\frac{1}{4}x^4-\frac{1}{3}x^3-x^2\right]_{-1}^{0}-\left[\frac{1}{4}x^4-\frac{1}{3}x^3-x^2\right]_{0}^{2}$$

$$=-\left(\frac{1}{4}+\frac{1}{3}-1\right)-\left(4-\frac{8}{3}-4\right)=\mathbf{\frac{37}{12}}$$



(2) $y=x^2+2x-3$, $y=-x^2+2x+3$

[Sol] From $x^2+2x-3=-x^2+2x+3$,

$$2x^2=6$$

$$x=\pm\sqrt{3}$$

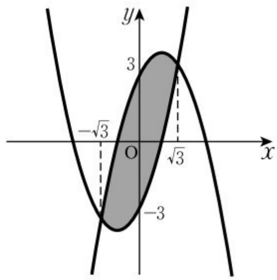
Therefore,

$$S=\int_{-\sqrt{3}}^{\sqrt{3}}[-x^2+2x+3-(x^2+2x-3)]dx$$

$$=2\int_{0}^{\sqrt{3}}(-2x^2+6)dx$$

$$=2\left[-\frac{2}{3}x^3+6x\right]_{0}^{\sqrt{3}}$$

$$=2(-2\sqrt{3}+6\sqrt{3})=\mathbf{8\sqrt{3}}$$



L 150b

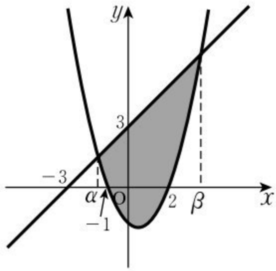
(3)　$y=x^2-x-2$,　　$y=x+3$

[Sol] From $x^2-x-2=x+3$,

$$x^2-2x-5=0 \quad \cdots ①$$

Let α and β (where $\alpha<\beta$) be the solutions of ①, so

$$\alpha+\beta=2,\ \alpha\beta=-5$$



Therefore,

$$\begin{aligned} S &= \int_{\alpha}^{\beta}[x+3-(x^2-x-2)]dx \\ &= -\int_{\alpha}^{\beta}(x^2-2x-5)dx \\ &= -\int_{\alpha}^{\beta}(x-\alpha)(x-\beta)dx \\ &= \frac{1}{6}(\beta-\alpha)^3 \end{aligned}$$

From $(\beta-\alpha)^2=(\alpha+\beta)^2-4\alpha\beta=24$,

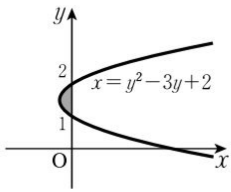
$$\beta-\alpha=2\sqrt{6}$$

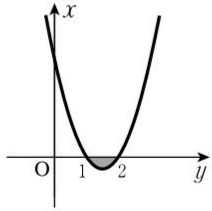
Therefore, $(\beta-\alpha)^3=48\sqrt{6}$

Thus, $\boldsymbol{S=8\sqrt{6}}$

Let's try this!

Find the area, S, of the shaded region shown in the top right figure.





[Sol] As shown in the bottom right figure, we can transform the graph onto a different grid by rearranging the x-axis to the y-axis, and the y-axis to the x-axis.

$$S=-\int_{1}^{2}(y^2-3y+2)dy$$

$$=-\left[\frac{1}{3}y^3-\frac{3}{2}y^2+2y\right]_1^2=\boxed{\mathbf{\frac{1}{6}}}$$

Answer: $\frac{1}{6}$

L 151 a

KUMON

Areas II

Ex.

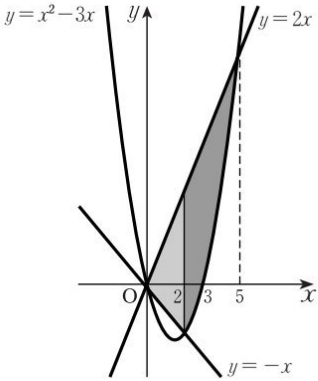
Find the area, S, enclosed by the parabola $y = x^2 - 3x$, the line $y = 2x$, and the line $y = -x$.

[Sol] Finding the x-coordinates of the points of intersection of $y = x^2 - 3x$ and $y = -x$:

From $x^2 - 3x = -x$,

$$x(x-2) = 0$$

$$x = 0,\ \boxed{\mathbf{2}}$$



Finding the x-coordinates of the points of intersection of $y = x^2 - 3x$ and $y = 2x$:

From $x^2 - 3x = 2x$,

$$x(x-5) = 0$$

$$x = 0,\ 5$$

Therefore,

$$S = \underbrace{\int_0^{\boxed{2}} [2x - (-x)]dx}_{\text{This represents the area of the light shaded region.}} + \underbrace{\int_{\boxed{2}}^{5} [2x - (x^2 - 3x)]dx}_{\text{This represents the area of the dark shaded region.}}$$

$$= \int_0^{\boxed{2}} 3x dx + \int_{\boxed{2}}^{5} (-x^2 + 5x) dx$$

$$= \left[\frac{3}{2}x^2\right]_0^{\boxed{2}} + \left[-\frac{1}{3}x^3 + \frac{5}{2}x^2\right]_{\boxed{2}}^{5}$$

$$= 6 + \left(-\frac{125}{3} + \frac{125}{2}\right) - \left(-\frac{8}{3} + 10\right)$$

$$= \boxed{\mathbf{\frac{39}{2}}}$$

Answers: in order 2, 2, 2, 2, 2, 2, 2, $\frac{39}{2}$

L151b

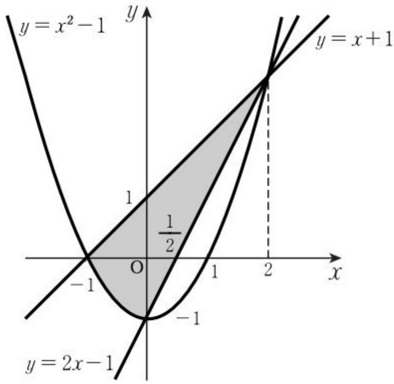
1. Find the area, S, enclosed by the parabola $y=x^2-1$, the line $y=2x-1$, and the line $y=x+1$.

[Sol] Finding the x-coordinates of the points of intersection of $y=x^2-1$ and $y=2x-1$:

From $x^2-1=2x-1$,

$x(x-2)=0$

$x=0,\ 2$



Finding the x-coordinates of the points of intersection of $y=x^2-1$ and $y=x+1$:

From $x^2-1=x+1$,

$x^2-x-2=0$

$(x-2)(x+1)=0$

$x=2,\ -1$

Finding the x-coordinates of the points of intersection of $y=2x-1$ and $y=x+1$:

From $2x-1=x+1$,

$x=2$

Therefore,

$$\begin{aligned}
\mathbf{S} &= \int_{-1}^{0}[x+1-(x^2-1)]dx+\int_{0}^{2}[x+1-(2x-1)]dx \\
&= \int_{-1}^{0}(-x^2+x+2)dx+\int_{0}^{2}(-x+2)dx \\
&= \left[-\frac{1}{3}x^3+\frac{1}{2}x^2+2x\right]_{-1}^{0}+\left[-\frac{1}{2}x^2+2x\right]_{0}^{2} \\
&= -\left(\frac{1}{3}+\frac{1}{2}-2\right)+(-2+4) \\
&= \mathbf{\frac{19}{6}}
\end{aligned}$$

L 152a KUMON

Areas II

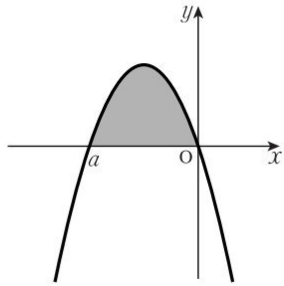
1. Find the value of a for which the area, S, enclosed by the parabola $y = ax - x^2$ and the x-axis is $\dfrac{9}{2}$.

[Sol] Finding the points of intersection with the x-axis:

From $ax - x^2 = -x(x-a) = 0$, $x = 0,\ a$

(i) When $a < 0$,

$$S = \int_a^0 (ax - x^2)\,dx$$
$$= \left[\frac{a}{2}x^2 - \frac{1}{3}x^3\right]_a^0$$
$$= -\frac{a^3}{6} = \frac{9}{2}$$

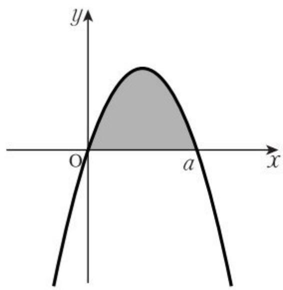


Therefore, $a^3 = -27$

Thus, $\boldsymbol{a = -3}$

(ii) When $a > 0$,

$$S = \int_0^a (ax - x^2)\,dx$$
$$= \left[\frac{a}{2}x^2 - \frac{1}{3}x^3\right]_0^a$$
$$= \frac{a^3}{6} = \frac{9}{2}$$



Therefore, $a^3 = 27$

Thus, $\boldsymbol{a = 3}$

From (i) and (ii),

$\boldsymbol{a = \pm 3}$

L 152b

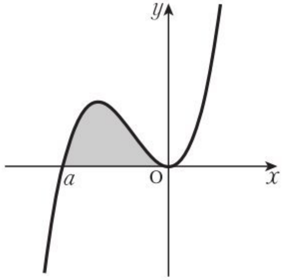
2. Find the value of a for which the area, S, enclosed by the curve $y=x^3-ax^2$ and the x-axis is $\frac{4}{3}$.

[Sol] Finding the points of intersection with the x-axis:

From $x^3-ax^2=x^2(x-a)=0$, $x=0$, a

(i) When $a<0$,

$$S=\int_a^0(x^3-ax^2)dx$$
$$=\left[\frac{1}{4}x^4-\frac{a}{3}x^3\right]_a^0$$
$$=\frac{a^4}{12}=\frac{4}{3}$$

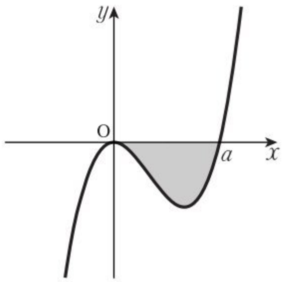


Therefore, $a^4=16$

Thus, $a=-2$

(ii) When $a>0$,

$$S=-\int_0^a(x^3-ax^2)dx$$
$$=-\left[\frac{1}{4}x^4-\frac{a}{3}x^3\right]_0^a$$
$$=\frac{a^4}{12}=\frac{4}{3}$$



Therefore, $a^4=16$

Thus, $a=2$

From (i) and (ii),

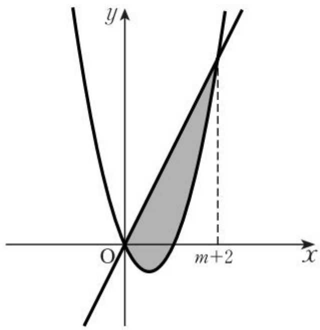
$\boldsymbol{a=\pm 2}$

L 153a KUMON

Areas II

1. The line $y=mx$, which passes through the origin, and the parabola $y=x^2-2x$ enclose a region whose area, S, is $\frac{32}{3}$. Find the value of m by the following steps. Assume that $m>0$.

(1) Find the x-coordinates of the points of intersection of $y=mx$ and $y=x^2-2x$.



[Sol] From $mx=x^2-2x$,

$$x^2-(m+2)x=x[x-(m+2)]=0$$

Therefore, $\boldsymbol{x=0,\ m+2}$

(2) Express the area, S, in terms of m.

[Sol] $$\boldsymbol{S}=\int_0^{m+2}[mx-(x^2-2x)]dx$$
$$=-\int_0^{m+2}[x^2-(m+2)x]dx$$
$$=-\int_0^{m+2}x[x-(m+2)]dx$$
$$=\frac{1}{6}(m+2-0)^3$$
$$\boldsymbol{=\frac{1}{6}(m+2)^3}$$

(3) Find the value of m for which the area, $S=\frac{32}{3}$.

[Sol] From $\frac{1}{6}(m+2)^3=\frac{32}{3}$,

$$(m+2)^3=64$$
$$m+2=4$$

Therefore, $\boldsymbol{m=2}$

L 153b

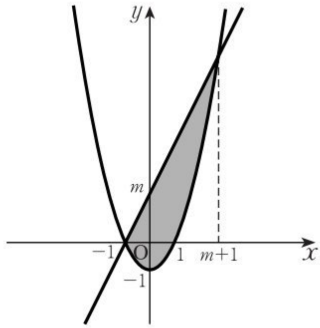
2. Find the value of m for which the line $y=m(x+1)$, which passes through point $(-1,\ 0)$, and the parabola $y=x^2-1$ enclose a region whose area, S, is $\frac{9}{2}$. Assume that $m>0$.

[Sol] Finding the x-coordinates of the points of intersection of $y=m(x+1)$ and $y=x^2-1$:

From $m(x+1)=x^2-1$,

$$x^2-mx-(m+1)=(x+1)[x-(m+1)]=0$$

$$x=-1,\ m+1$$



Therefore,

$$\begin{aligned} S &= \int_{-1}^{m+1}[m(x+1)-(x^2-1)]dx \\ &= \int_{-1}^{m+1}(-x^2+mx+m+1)dx \\ &= -\int_{-1}^{m+1}(x^2-mx-m-1)dx \\ &= -\int_{-1}^{m+1}(x+1)[x-(m+1)]dx \\ &= \frac{1}{6}(m+2)^3=\frac{9}{2} \end{aligned}$$

Therefore,

$$(m+2)^3=27$$

$$m+2=3$$

Thus, $\boldsymbol{m=1}$

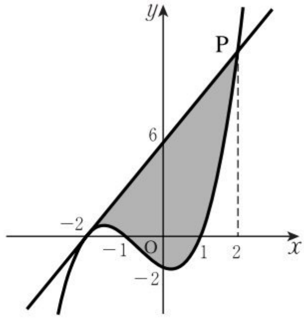
L 154a

KUMON

Areas II

1. Find the area, S, enclosed by the curve $y=x^3+2x^2-x-2$ and the line tangent to the curve at point $(-2,\ 0)$ by the following steps.

(1) Find the equation of the tangent.



[Sol] Let $f(x)=x^3+2x^2-x-2$

$$f'(x)=3x^2+4x-1,\ f'(-2)=3$$

Therefore,

$$\begin{aligned} y &= 3[x-(-2)] \\ &= 3(x+2) \\ &= 3x+6 \end{aligned}$$

Since the line has gradient 3 and passes through point $(-2,\ 0)$.

Thus, $\boldsymbol{y=3x+6}$

(2) Find the x-coordinate of the point of intersection, P, other than the point of tangency.

[Sol] $x^3+2x^2-x-2=3x+6$

$$x^3+2x^2-4x-8=0$$

$$(x+2)^2(x-2)=0$$

Therefore, $\boldsymbol{x=2}$

(3) Find the area, S, enclosed between the curve and the tangent.

[Sol] $$\begin{aligned} \boldsymbol{S} &= \int_{-2}^{2}[3x+6-(x^3+2x^2-x-2)]dx \\ &= \int_{-2}^{2}(-x^3-2x^2+4x+8)dx \\ &= 2\int_{0}^{2}(-2x^2+8)dx \\ &= 2\left[-\frac{2}{3}x^3+8x\right]_0^2 \\ &= 2\left(-\frac{16}{3}+16\right)=\boldsymbol{\frac{64}{3}} \end{aligned}$$

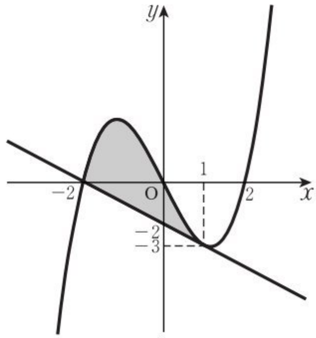
2. Find the area, S, enclosed by the curve $y=x^3-4x$ and the line tangent to the curve at point $(1, -3)$.

[Sol] Let $f(x)=x^3-4x$

$$f'(x)=3x^2-4,\ f'(1)=-1$$

Therefore, since the line passes through point $(1, -3)$,

$$y=-(x-1)-3$$
$$=-x-2$$



Finding the x-coordinate of the point of intersection, other than the point of tangency,

$$x^3-4x=-x-2$$
$$x^3-3x+2=0$$
$$(x-1)^2(x+2)=0$$
$$x=-2$$

Therefore,

$$\boldsymbol{S}=\int_{-2}^{1}[x^3-4x-(-x-2)]dx$$
$$=\int_{-2}^{1}(x^3-3x+2)dx$$
$$=\left[\frac{1}{4}x^4-\frac{3}{2}x^2+2x\right]_{-2}^{1}$$
$$=\left(\frac{1}{4}-\frac{3}{2}+2\right)-(4-6-4)$$
$$\boldsymbol{=\frac{27}{4}}$$

L 155a KUMON

Areas II

1. Given that the curve $y=x^3$ has a tangent that passes through point $(0,\ 2)$, find the area enclosed by the curve and the line by the following steps.

(1) Find the equation of the tangent.

[Sol] Let $f(x)=x^3$, and let the point of tangency be $(t,\ t^3)$.

$f'(x)=3x^2,\ f'(t)=3t^2$

The equation of the tangent is:

$y=3t^2(x-t)+t^3$

$=3t^2x-2t^3 \qquad \cdots①$

Since the tangent has gradient $3t^2$ and passes through point $(t,\ t^3)$.

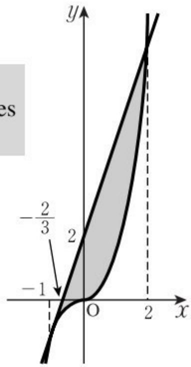
Since the line passes through $(0,\ 2)$,

from $2=-2t^3$,

$t=-1$

Therefore, $\boldsymbol{y=3x+2}$

Substituting $t=-1$ into ①.



(2) Find the x-coordinates of the points of intersection between the curve and the tangent.

[Sol] $x^3=3x+2$

$x^3-3x-2=0$

$(x+1)^2(x-2)=0$

$\boldsymbol{x=-1,\ 2}$

(3) Find the area, S, enclosed between the curve and the tangent.

[Sol] $\boldsymbol{S}=\displaystyle\int_{-1}^{2}(3x+2-x^3)dx$

$=\left[-\dfrac{1}{4}x^4+\dfrac{3}{2}x^2+2x\right]_{-1}^{2}$

$=(-4+6+4)-\left(-\dfrac{1}{4}+\dfrac{3}{2}-2\right)=\boldsymbol{\dfrac{27}{4}}$

L 155b

2. Find the area, S, enclosed by the curve $y=x^3-x^2-2x$ and the tangent that passes through point $(0, -1)$.

[Sol] Let $f(x)=x^3-x^2-2x$, and let the point of tangency be $(t,\ t^3-t^2-2t)$.

$$f'(x)=3x^2-2x-2,\ \ f'(t)=3t^2-2t-2$$

The equation of the tangent is:

$$y=(3t^2-2t-2)(x-t)+t^3-t^2-2t$$
$$=(3t^2-2t-2)x-2t^3+t^2$$

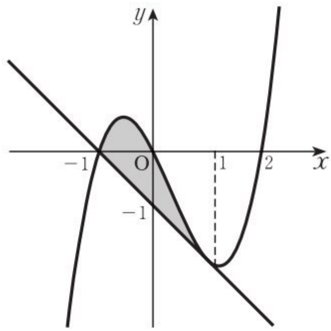
Since the line passes through $(0,\ -1)$,

from $-1=-2t^3+t^2$,

$$2t^3-t^2-1=(t-1)(2t^2+t+1)=0$$

$2t^2+t+1\neq 0$ and $t=1$

Therefore, $y=-x-1$



Finding the x-coordinates of the common points between the curve and the tangent,

$$x^3-x^2-2x=-x-1$$
$$x^3-x^2-x+1=(x-1)^2(x+1)=0$$
$$x=-1,\ 1$$

Therefore,

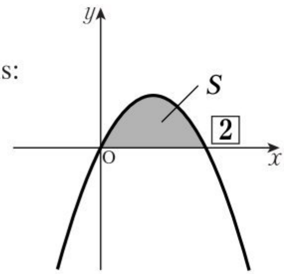
$$\mathbf{S}=\int_{-1}^{1}[x^3-x^2-2x-(-x-1)]dx$$
$$=\int_{-1}^{1}(x^3-x^2-x+1)dx$$
$$=2\int_{0}^{1}(-x^2+1)dx$$
$$=2\left[-\frac{1}{3}x^3+x\right]_0^1$$
$$=2\left(-\frac{1}{3}+1\right)=\mathbf{\frac{4}{3}}$$

L 156a KUMON

Areas II

1. By applying the following steps, find the value of m for which the area enclosed by the parabola $y=2x-x^2$ and the x-axis is divided into two equal halves by the line $y=mx$.

(1) Find the area, S, enclosed by $y=2x-x^2$ and the x-axis.



[Sol] Finding the x-coordinates of the points of intersection between $y=2x-x^2$ and the x-axis:

From $2x-x^2=x(2-x)=0$, $x=0$, 2

Therefore,

$$\boldsymbol{S}=\int_0^2(2x-x^2)dx=-\int_0^2 x(x-2)dx$$

$$=\frac{1}{6}(2-0)^3 = \boldsymbol{\frac{4}{3}}$$

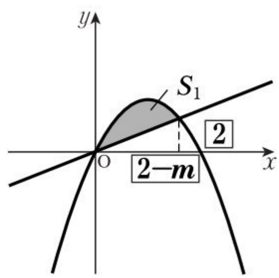
(2) Find the area, S_1, enclosed by $y=2x-x^2$ and $y=mx$.

[Sol] Finding the x-coordinates of the points of intersection between $y=2x-x^2$ and $y=mx$:

From $2x-x^2=mx$, $x=0$, $2-m$

Therefore, the range of values of m is $0<2-m<\boxed{2}$.

Thus, $\boxed{0}<m<\boxed{2}$ $\cdots$①



$$S_1=\int_0^{2-m}(2x-x^2-mx)dx$$

$$=-\int_0^{2-m}x[x-(2-m)]dx$$

$$=\frac{1}{6}[(2-m)-0]^3=\boldsymbol{\frac{1}{6}(2-m)^3}$$

(3) Find the value of m for which $S=2S_1$.

[Sol] From (1) and (2),

$$\frac{4}{3}=2\cdot\frac{1}{6}(2-m)^3$$

Therefore, $(2-m)^3=4$

Thus, the value of m that satisfies condition ① is $m=\boxed{2}-\sqrt[3]{\boxed{4}}$.

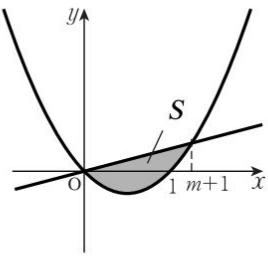
L 156b

2. Find the value of m for which the area, S, enclosed by the parabola $y=x^2-x$ and the line $y=mx$ is divided into two equal halves by the x-axis. Assume that $m>0$.

[Sol] Finding the x-coordinates of the points of intersection between $y=x^2-x$ and $y=mx$:

From $x^2-x=mx$, $\quad x=0,\ m+1$

Therefore,



$$S=\int_0^{m+1}[mx-(x^2-x)]dx$$

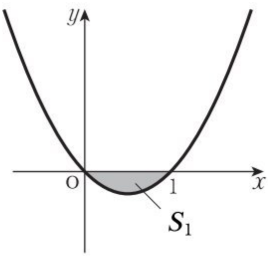
$$=-\int_0^{m+1}x[x-(m+1)]dx$$

$$=\frac{1}{6}[(m+1)-0]^3=\frac{1}{6}(m+1)^3$$

Finding the points of intersection between $y=x^2-x$ and the x-axis:

From $x^2-x=x(x-1)=0$, $\quad x=0,\ 1$

Therefore,



$$S_1=-\int_0^1(x^2-x)dx$$

$$=-\int_0^1 x(x-1)dx$$

$$=\frac{1}{6}(1-0)^3=\frac{1}{6}$$

From $S=2S_1$,

$$\frac{1}{6}(m+1)^3=2\cdot\frac{1}{6}$$

$$(m+1)^3=2$$

Therefore, $\boldsymbol{m=-1+\sqrt[3]{2}}$

L 157a KUMON

Areas II

1. When the area enclosed by the curve $y = x(a-x)$ and the x-axis is S, and the area enclosed by the curve $y = x^2(a-x)$ and the x-axis is S_1, find the value of a for which $S = 2S_1$. Assume that $0 < a \leqslant 1$.

[Sol]

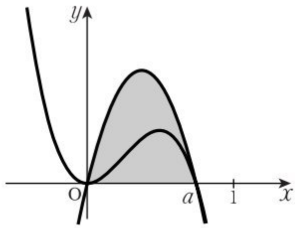
Finding the points of intersection between $y = x(a-x)$ and the x-axis:

From $x(a-x) = 0$, $x = 0, a$

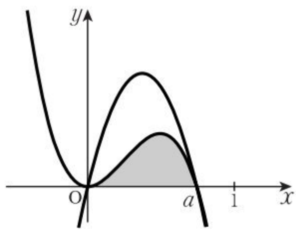
Finding the points of intersection between $y = x^2(a-x)$ and the x-axis:

From $x^2(a-x) = 0$, $x = 0, a$

Therefore,



$$S = \int_0^a x(a-x)dx$$
$$= \int_0^a (-x^2 + ax)dx$$
$$= \left[-\frac{1}{3}x^3 + \frac{a}{2}x^2\right]_0^a = -\frac{a^3}{3} + \frac{a^3}{2} = \frac{a^3}{6}$$



$$S_1 = \int_0^a x^2(a-x)dx$$
$$= \int_0^a (-x^3 + ax^2)dx$$
$$= \left[-\frac{1}{4}x^4 + \frac{a}{3}x^3\right]_0^a = -\frac{a^4}{4} + \frac{a^4}{3} = \frac{a^4}{12}$$

From $S = 2S_1$,

$$\frac{a^3}{6} = 2 \cdot \frac{a^4}{12}$$
$$a^3(1-a) = 0$$
$$a = 0, 1$$

Since $0 < a \leqslant 1$,

$$\boldsymbol{a = 1}$$

L 157b

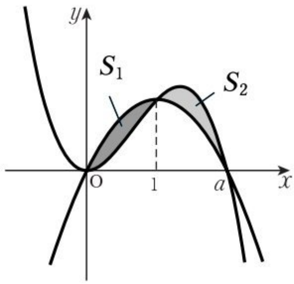
2. Find the value of a for which the two regions, S_1 and S_2, enclosed by the curve $y=x(a-x)$ and the curve $y=x^2(a-x)$, have equal areas.
Assume that $a>1$.

[Sol] Finding the x-coordinates of the points of intersection between $y=x(a-x)$ and $y=x^2(a-x)$:

From $x(a-x)=x^2(a-x)$,

$$x(a-x)(x-1)=0$$

$$x=0,\ 1,\ a$$



Therefore,

$$S_1=\int_0^1[x(a-x)-x^2(a-x)]dx$$

$$=\int_0^1[x^3-(a+1)x^2+ax]dx$$

$$=\left[\frac{1}{4}x^4-\frac{a+1}{3}x^3+\frac{a}{2}x^2\right]_0^1$$

$$=\frac{1}{4}-\frac{a+1}{3}+\frac{a}{2}=\frac{a}{6}-\frac{1}{12}$$

$$S_2=\int_1^a[x^2(a-x)-x(a-x)]dx$$

$$=\int_1^a[-x^3+(a+1)x^2-ax]dx$$

$$=\left[-\frac{1}{4}x^4+\frac{a+1}{3}x^3-\frac{a}{2}x^2\right]_1^a$$

$$=\left(-\frac{a^4}{4}+\frac{a+1}{3}a^3-\frac{a^3}{2}\right)-\left(-\frac{1}{4}+\frac{a+1}{3}-\frac{a}{2}\right)=\frac{a^4}{12}-\frac{a^3}{6}+\frac{a}{6}-\frac{1}{12}$$

From $S_1=S_2$,

$$\frac{a}{6}-\frac{1}{12}=\left(\frac{a^4}{12}-\frac{a^3}{6}+\frac{a}{6}-\frac{1}{12}\right)$$

$$a^3(a-2)=0$$

$$a=0,\ 2$$

Since $a>1$,

$$\boldsymbol{a=2}$$

L 158a

Areas II

Ex.

Find the value of m that gives the minimum value of the area, S, enclosed by the parabola $y=x^2+x-1$ and the line $y=mx$.

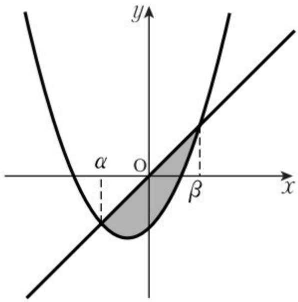
[Sol] Finding the x-coordinates of the points of intersection between $y=x^2+x-1$ and $y=mx$:

From $x^2+x-1=mx$,

$$x^2+(1-m)x-1=0 \quad \cdots ①$$

If the two roots of ① are α and β (where $\alpha<\beta$),

$$\alpha+\beta=\boxed{\boldsymbol{m-1}},\ \alpha\beta=\boxed{\mathbf{-1}}$$



$$S=\int_\alpha^\beta [mx-(x^2+x-1)]dx$$

$$=-\int_\alpha^\beta [x^2+(1-m)x-1]dx$$

$$=\frac{1}{6}(\beta-\alpha)^3$$

$$=\frac{1}{6}\left[\sqrt{(\alpha+\beta)^2-4\alpha\beta}\right]^3$$

$$=\frac{1}{6}\left[\sqrt{(m-\boxed{\mathbf{1}})^2+\boxed{\mathbf{4}}}\right]^3$$

There is a minimum value when $(m-\boxed{})^2$ is zero.

Therefore, $m=\boxed{\mathbf{1}}$

Answers: in order $m-1$, -1, 1, 4, 1

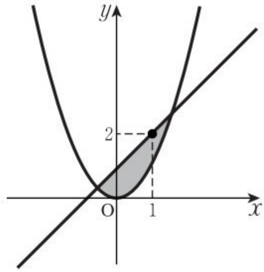
L 158b

1. Given a region enclosed by the parabola $y=x^2$ and the line $y=m(x-1)+2$, which passes through point $(1,\ 2)$, find the value of m that gives the minimum value of the area, S, of the enclosed region.

[Sol] Finding the x-coordinates of the points of intersection between $y=x^2$ and $y=m(x-1)+2$:

From $x^2=m(x-1)+2$,

$$x^2-mx+m-2=0 \quad \cdots①$$



If the two roots of ① are α and β (where $\alpha<\beta$),

$$\alpha+\beta=m,\ \alpha\beta=m-2$$

$$S=\int_\alpha^\beta [m(x-1)+2-x^2]dx$$

$$=-\int_\alpha^\beta (x^2-mx+m-2)dx$$

$$=\frac{1}{6}(\beta-\alpha)^3$$

$$=\frac{1}{6}\left[\sqrt{(\alpha+\beta)^2-4\alpha\beta}\right]^3$$

$$=\frac{1}{6}\left[\sqrt{m^2-4(m-2)}\right]^3$$

$$=\frac{1}{6}\left[\sqrt{(m-2)^2+4}\right]^3$$

Therefore, $\boldsymbol{m=2}$

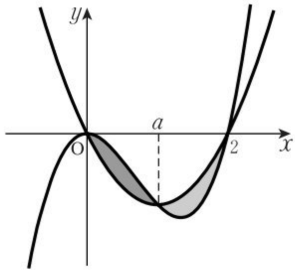
L 159a

KUMON

Areas II

1. Find the value of a for which the area, $S(a)$, enclosed by the curves $y=x^2(x-2)$ and $y=ax(x-2)$, is at a minimum. Assume that $0<a<2$.

[Sol] Finding the x-coordinates of the points of intersection between $y=x^2(x-2)$ and $y=ax(x-2)$:



From $x^2(x-2)=ax(x-2)$,

$x(x-2)(x-a)=0$

$x=0,\ a,\ 2$

Therefore,

$$S(a)=\int_0^a[x^2(x-2)-ax(x-2)]dx+\int_a^2[ax(x-2)-x^2(x-2)]dx$$

$$=\int_0^a[x^3-(a+2)x^2+2ax]dx+\int_a^2[-x^3+(a+2)x^2-2ax]dx$$

$$=\left[\frac{1}{4}x^4-\frac{a+2}{3}x^3+ax^2\right]_0^a+\left[-\frac{1}{4}x^4+\frac{a+2}{3}x^3-ax^2\right]_a^2$$

$$=\boldsymbol{-\frac{a^4}{6}+\frac{2}{3}a^3-\frac{4}{3}a+\frac{4}{3}}$$

$$S'(a)=-\frac{2}{3}a^3+2a^2-\frac{4}{3}=\boldsymbol{-\frac{2}{3}(a-1)(a^2-2a-2)}$$

From $S'(a)=0$,

$a=1,\ 1\pm\sqrt{3}$ ☞ $0<a<2$ so $1\pm\sqrt{3}$ are extraneous solutions.

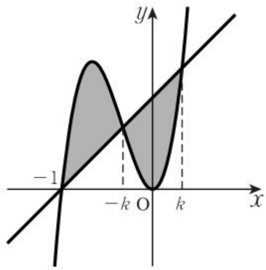
a	**0**	…	**1**	…	**2**
$S'(a)$		−	**0**	+	
$S(a)$		↘	**relative minimum**	↗	

Therefore, $a=\boxed{1}$

L 159b

2. Find the value of k for which the area, $S(k)$, enclosed by the curve $y=x^2(x+1)$ and the line $y=k^2(x+1)$, is at a minimum. Assume that $0<k<1$.

[Sol] Finding the x-coordinates of the points of intersection between $y=x^2(x+1)$ and $y=k^2(x+1)$:



From $x^2(x+1)=k^2(x+1)$,

$$(x+1)(x+k)(x-k)=0$$

$$x=-1,\ -k,\ k$$

Therefore,

$$S(k)=\int_{-1}^{-k}[x^2(x+1)-k^2(x+1)]dx+\int_{-k}^{k}[k^2(x+1)-x^2(x+1)]dx$$

$$=\int_{-1}^{-k}(x^3+x^2-k^2x-k^2)dx+\int_{-k}^{k}(-x^3-x^2+k^2x+k^2)dx$$

$$=\int_{-1}^{-k}(x^3+x^2-k^2x-k^2)dx+2\int_{0}^{k}(-x^2+k^2)dx$$

$$=\left[\frac{1}{4}x^4+\frac{1}{3}x^3-\frac{k^2}{2}x^2-k^2x\right]_{-1}^{-k}+2\left[-\frac{1}{3}x^3+k^2x\right]_0^k$$

$$=\left(\frac{k^4}{4}-\frac{k^3}{3}-\frac{k^4}{2}+k^3\right)-\left(\frac{1}{4}-\frac{1}{3}-\frac{k^2}{2}+k^2\right)+2\left(-\frac{k^3}{3}+k^3\right)$$

$$=-\frac{k^4}{4}+2k^3-\frac{k^2}{2}+\frac{1}{12}$$

From $S'(k)=-k^3+6k^2-k=-k(k^2-6k+1)=0$,

$$k=0,\ 3\pm2\sqrt{2}$$

k	0	$\cdots$	$3-2\sqrt{2}$	$\cdots$	1
$S'(k)$		$-$	0	$+$	
$S(k)$		↘	relative minimum	↗	

Therefore, $\boldsymbol{k=3-2\sqrt{2}}$

L 160a KUMON

Areas II

1. Find the equation of any line that passes through point $(0,\ 1)$ such that the area, S, enclosed by this line and the parabola $y=x^2$ is $\dfrac{5\sqrt{5}}{6}$.

[Sol] Let the equation of the line be $y=mx+1$.

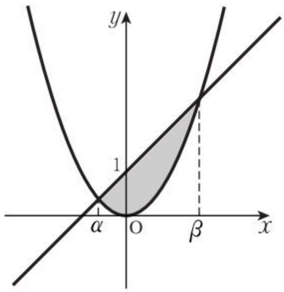
Finding the x-coordinates of the points of intersection between $y=x^2$ and $y=mx+1$:

From $x^2=mx+1$,

$$x^2-mx-1=0$$

If the two roots of the equation are α and β (where $\alpha<\beta$),

$$\alpha+\beta=m,\ \alpha\beta=-1$$



Therefore,

$$S=\int_\alpha^\beta[(mx+1)-x^2]dx$$

$$=-\int_\alpha^\beta(x^2-mx-1)dx$$

$$=\frac{1}{6}(\beta-\alpha)^3$$

$$=\frac{1}{6}\left[\sqrt{(\alpha+\beta)^2-4\alpha\beta}\right]^3$$

$$=\frac{1}{6}\left(\sqrt{m^2+4}\right)^3=\frac{5\sqrt{5}}{6}$$

Therefore,

$$\left(\sqrt{m^2+4}\right)^3=5\sqrt{5}$$

$$m^2+4=5$$

$$m^2=1$$

Therefore, $m=\pm1$

Thus, $\boldsymbol{y=x+1,\ \ y=-x+1}$

L 160b

2. Find the area, *S*, enclosed by the curve $y=x^3-7x$ and the line tangent to the curve at point $(-1,\ 6)$.

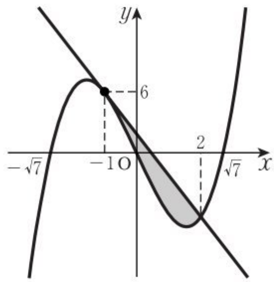
[Sol] Let $f(x)=x^3-7x$

$$f'(x)=3x^2-7,\ f'(-1)=-4$$

Therefore, since the line passes through $(-1,\ 6)$,

$$\begin{aligned} y&=-4(x+1)+6 \\ &=-4x+2 \end{aligned}$$

Finding the x-coordinate of the point of intersection, other than the point of tangency:



$$x^3-7x=-4x+2$$

$$x^3-3x-2=0$$

$$(x+1)^2(x-2)=0$$

$$x=2$$

Therefore,

$$\begin{aligned} \boldsymbol{S}&=\int_{-1}^{2}[-4x+2-(x^3-7x)]dx \\ &=\int_{-1}^{2}(-x^3+3x+2)dx \\ &=\left[-\frac{1}{4}x^4+\frac{3}{2}x^2+2x\right]_{-1}^{2} \\ &=(-4+6+4)-\left(-\frac{1}{4}+\frac{3}{2}-2\right) \\ &=\boldsymbol{\frac{27}{4}} \end{aligned}$$

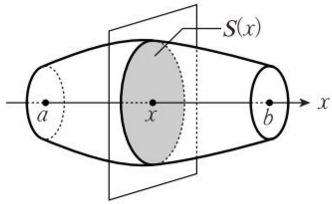
L 161 a

KUMON

Volumes

If $S(x)$ is a function of x representing the cross-sectional area of a given solid, cut by a plane perpendicular to the x-axis, then the volume for $a \leqslant x \leqslant b$ is given by the following formula:

$$\boxed{V = \int_a^b S(x)\,dx}$$



Ex.

Find the volume, V, of a cone with base radius, r, and height, h.

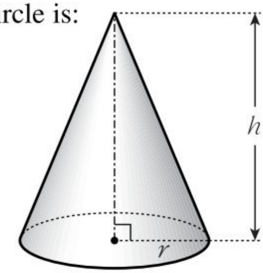
[Sol] Let the vertex be the origin, O; let the altitude (the perpendicular line dropped down from O to the base) be the x-axis; and let the cross-sectional area be $S(x)$.

If the radius of the cross-sectional (shaded) circle is r',

from $r' : r = x : h$, $\quad r' = \dfrac{r}{h}x$

Therefore, the area of the cross-sectional circle is:

$$S(x) = \pi\left(\frac{r}{h}x\right)^2 = \frac{\pi r^2}{h^2}x^2$$



The range of x-values is:

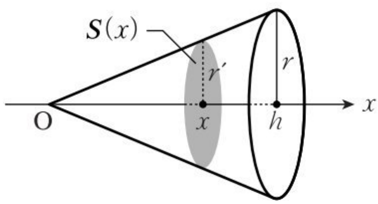
$$0 \leqslant x \leqslant h$$

Therefore,

$$V = \int_0^h \frac{\pi r^2}{h^2}x^2\,dx$$

$$= \frac{\pi r^2}{h^2}\left[\boxed{\frac{1}{3}x^3}\right]_0^h$$

$$= \boxed{\frac{1}{3}\pi r^2 h}$$



Answers: in order $\frac{1}{3}x^3$, $\frac{1}{3}\pi r^2 h$

Definition: The perpendicular distance from the vertex to the plane of the base is the ***altitude*** of the cone.

L161b

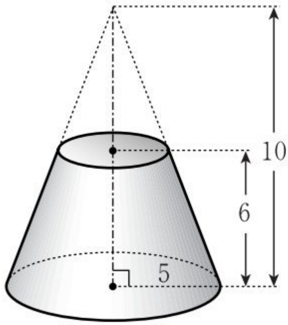
1. When a cone with base radius 5 and overall height 10 is divided into two parts at height 6, find the volume of the shaded region shown below.

[Sol] Let the vertex of the original cone be the origin, O; let the altitude (the perpendicular line dropped down from O to the base) be the x-axis; and let the cross-sectional area be $S(x)$.

If the radius of the cross-sectional (shaded) circle is r',

from $r' : \boxed{\mathbf{5}} = x : \boxed{\mathbf{10}}$,

$$r' = \boxed{\frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{x}}$$



Therefore,

$$S(x) = \pi\left(\frac{1}{2}x\right)^2 = \frac{\boldsymbol{\pi}}{\mathbf{4}}\boldsymbol{x}^2$$

The range of x-values is:

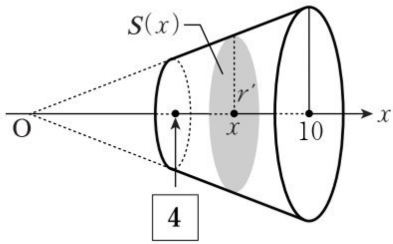
$$\boxed{\mathbf{4}} \leqslant x \leqslant 10$$

Therefore,

$$V = \int_4^{10} \frac{\pi}{4}x^2 dx$$

$$= \frac{\pi}{4}\left[\frac{1}{3}x^3\right]_4^{10}$$

$$= \mathbf{78\pi}$$



Definition: The part of a solid between two parallel planes cutting the solid is called a ***frustum***.

L 162a KUMON

Volumes

1. Find the volume, V, of a regular pyramid with four faces, whose square base has side length a, and whose height is h.

[Sol] Let the vertex be the origin, O; let the altitude (the perpendicular line dropped down from O to the base) be the x-axis; and let the cross-sectional area be $S(x)$.

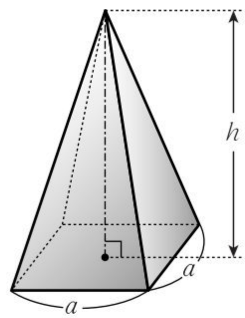
If the side length of the cross-sectional square is a',

from $a' : a = \boxed{\boldsymbol{x}} : \boxed{\boldsymbol{h}}$,

$$a' = \frac{\boldsymbol{a}}{\boldsymbol{h}}\boldsymbol{x}$$

Therefore, the area of the cross-sectional square is:

$$S(x) = \left(\frac{a}{h}x\right)^2 = \frac{\boldsymbol{a}^2}{\boldsymbol{h}^2}\boldsymbol{x}^2$$



The range of x-values is:

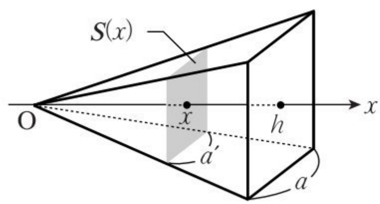
$$0 \leqslant x \leqslant h$$

Therefore,

$$\boldsymbol{V} = \int_0^h \frac{a^2}{h^2}x^2dx$$

$$= \frac{a^2}{h^2}\left[\frac{1}{3}x^3\right]_0^h$$

$$= \frac{\mathbf{1}}{\mathbf{3}}\boldsymbol{a}^2\boldsymbol{h}$$



2. When a regular pyramid with four faces, whose square base has side length 6 and whose height is 9, is divided into two parts at height 6 by a plane parallel to the base, find the volume, V, of the shaded region shown below.

[Sol] Let the vertex of the original pyramid be the origin, O; let the altitude (the perpendicular line dropped down from O to the base) be the x-axis; and let the cross-sectional area be $\boldsymbol{S}(x)$.

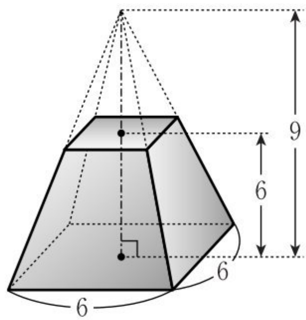
If the side length of the cross-sectional square is a',

from $a' : 6 = x : 9$,

$$a' = \frac{2}{3}x$$

Therefore, the area of the cross-sectional square is:

$$\boldsymbol{S}(x) = \left(\frac{2}{3}x\right)^2 = \frac{4}{9}x^2$$



The range of x-values is:

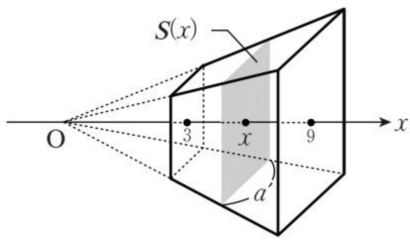
$$3 \leqslant x \leqslant 9$$

Therefore,

$$\boldsymbol{V} = \int_3^9 \frac{4}{9}x^2 dx$$

$$= \frac{4}{9}\left[\frac{1}{3}x^3\right]_3^9$$

$$\mathbf{= 104}$$



L 163a KUMON

Volumes

1. Find the volume, V, of a sphere with radius a.

[Sol] We place the origin at the centre of the sphere and set the diameter as the x-axis. When the sphere is cut by a plane perpendicular to the x-axis, let the cross-sectional area be $S(x)$.

If the radius of the cross-sectional circle is r,
from $x^2+r^2=a^2$, $\quad r=\sqrt{a^2-x^2}$

Therefore, the area of the cross-sectional circle is:

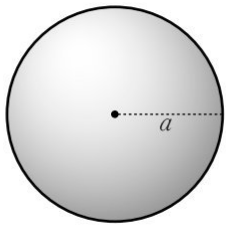
$$S(x)=\pi r^2=\pi(a^2-x^2)$$

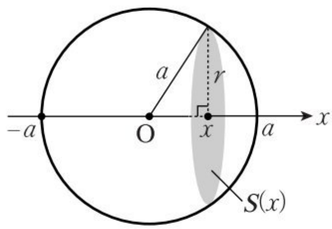
The range of x-values is:

$$-a \leqslant x \leqslant a$$

Therefore,

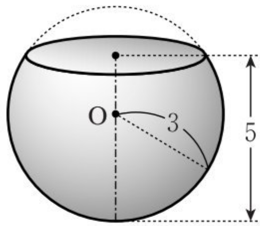
$$\begin{aligned} V &= \int_{-a}^{a} \pi(a^2-x^2)dx \\ &= 2\pi\int_0^a (a^2-x^2)dx \\ &= 2\pi\left[a^2x-\frac{1}{3}x^3\right]_0^a \\ &= \boldsymbol{\frac{4}{3}\pi a^3} \end{aligned}$$





2. Given that a sphere with radius 3 and centre at O is divided into two parts by the plane perpendicular to the line which passes through O at height 5, find the volume, V, of the shaded region below.

[Sol] Since the centre of the original sphere is at O, let the line that is perpendicular to the cross-sectional area be the x-axis; and let the cross-sectional area be $S(x)$.



If the radius of the cross-sectional circle is r,

$$r=\sqrt{3^2-x^2}=\sqrt{9-x^2}$$

Therefore,

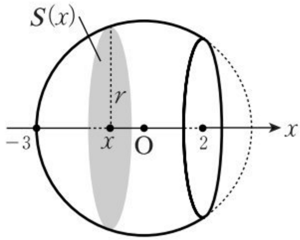
the area of the cross-sectional circle is:

$$S(x)=\pi(\sqrt{9-x^2})^2=\pi(9-x^2)$$

The range of x-values is:

$$-3\leqslant x\leqslant 2$$

Therefore,



$$\begin{aligned}V&=\int_{-3}^{2}\pi(9-x^2)dx\\&=\pi\int_{-3}^{2}(9-x^2)dx\\&=\pi\left[9x-\frac{1}{3}x^3\right]_{-3}^{2}\\&=\mathbf{\frac{100}{3}\pi}\end{aligned}$$

L 164a KUMON

Volumes

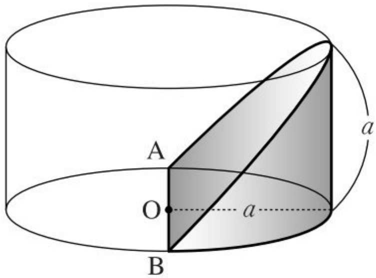
1. There is a right circular cylinder with base radius a and height a. When we cut the right cylinder into two parts, along the plane that includes the diameter AB and forms a $45°$ angle of inclination with the base, find the volume of the smaller of the two solids.

[Sol] Let the centre of the base be the origin, and place the diameter AB along the x-axis.

We cut this piece with a plane that is perpendicular to the x-axis at point P, whose x-coordinate is x. The cross-section is a right isosceles triangle.

From $\mathrm{PQ}^2+\mathrm{PO}^2=\mathrm{QO}^2$,

$$\mathrm{PQ}=\sqrt{a^2-x^2}$$



Since $\mathrm{QR}=\mathrm{PQ}$,

the area of the cross-section is:

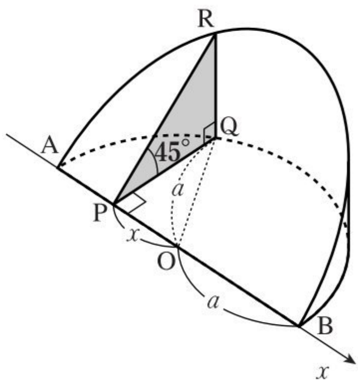
$$S(x)=\frac{1}{2}\mathrm{PQ}\cdot\mathrm{QR}$$

$$=\frac{1}{2}\mathrm{PQ}^2$$

$$=\frac{1}{2}(a^2-x^2)$$

The range of x-values is:

$$-a \leqslant x \leqslant \boxed{\boldsymbol{a}}$$



Therefore,

$$V=\int_{-a}^{a}\frac{1}{2}(a^2-x^2)dx$$

$$=\frac{1}{2}\cdot 2\int_{0}^{a}(a^2-x^2)dx$$

$$=\left[a^2x-\frac{1}{3}x^3\right]_0^a$$

$$=\boldsymbol{\frac{2}{3}a^3}$$

L 164b

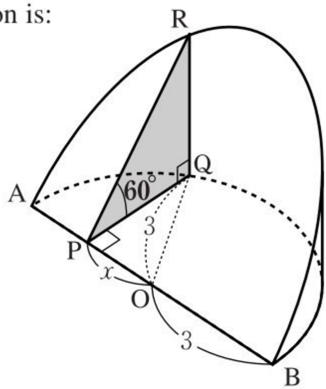
2. There is a right circular cylinder with base radius 3 and height 6. When we cut the right cylinder into two parts, along the plane that includes the diameter AB and forms a $60°$ angle of inclination with the base, find the volume of the smaller of the two solids.

[Sol] Let the centre of the base be the origin, and place the diameter AB along the x-axis.

$$\mathrm{PQ} = \sqrt{3^2 - x^2} = \sqrt{9 - x^2}$$

$$\mathrm{QR} = \sqrt{3}\,\mathrm{PQ}$$

Therefore, the area of the cross-section is:



$$S(x) = \frac{1}{2}\mathrm{PQ}\cdot\mathrm{QR}$$

$$= \frac{\sqrt{3}}{2}\mathrm{PQ}^2$$

$$= \boldsymbol{\frac{\sqrt{3}}{2}(9 - x^2)}$$

The range of x-values is:

$$-3 \leqslant x \leqslant 3$$

Therefore,

$$\boldsymbol{V} = \int_{-3}^{3} \frac{\sqrt{3}}{2}(9 - x^2)\,dx$$

$$= \frac{\sqrt{3}}{2}\cdot 2\int_{0}^{3}(9 - x^2)\,dx$$

$$= \sqrt{3}\left[9x - \frac{1}{3}x^3\right]_0^3$$

$$= \sqrt{3}\,(27 - 9)$$

$$\boldsymbol{= 18\sqrt{3}}$$

L 165a KUMON

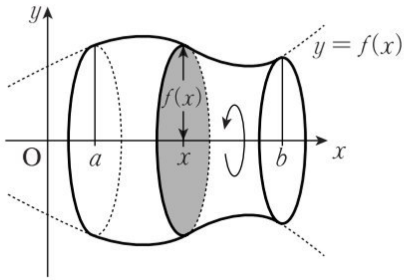
Volumes

The solid formed by rotating a line or curve around an axis is called a ***solid of revolution***. Let V be the volume of the solid formed by rotating the region enclosed by the function $y=f(x)$, the x-axis, line $x=a$, and line $x=b$. Since the cross-section is a circle, its area is:

$$S(x)=\pi y^2=\pi[f(x)]^2$$

Therefore, the volume, V, of the solid of revolution for $a \leqslant x \leqslant b$ is:

$$\boxed{V=\int_a^b \pi y^2 dx=\pi\int_a^b [f(x)]^2 dx}$$



Ex.

The solid formed by rotating the region enclosed by the line $y=\frac{1}{2}x$, the x-axis and the line $x=6$ around the x-axis is a circular cone with base radius 3 and height 6. Find the volume, V, of this circular cone.

[Sol] From the graph,

$$0 \leqslant x \leqslant 6$$

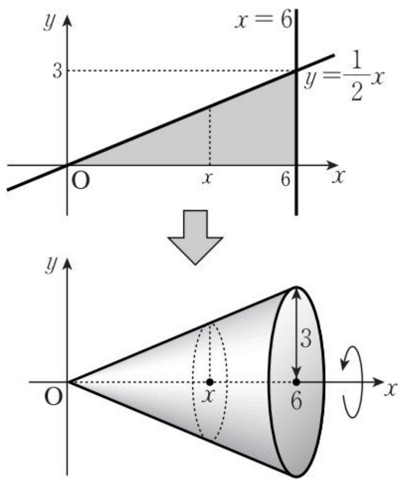
Therefore,

$$V=\pi\int_0^6\left(\frac{1}{2}x\right)^2 dx$$

$$=\pi\int_0^6 \frac{1}{4}x^2 dx$$

$$=\frac{\pi}{4}\left[\boxed{\frac{1}{3}x^3}\right]_0^6$$

$$=\boxed{18\pi}$$



Answers: in order $\frac{1}{3}x^3$, 18π

L 165b

1. Find the volume, V, of the solid (the frustum of a cone) formed by rotating the region enclosed by the line $y=\frac{1}{3}x$, the x-axis, the line $x=6$, and the line $x=9$ about the x-axis.

[Sol] From the graph,

$$\boxed{\mathbf{6}} \leqslant x \leqslant \boxed{\mathbf{9}}$$

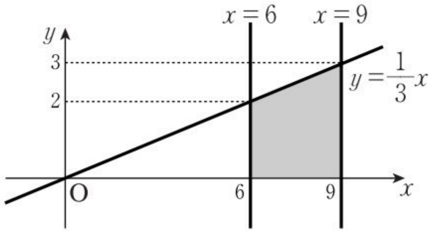
Therefore,

$$V=\pi\int_6^9\left(\frac{1}{3}x\right)^2dx$$

$$=\pi\int_6^9\frac{1}{9}x^2dx$$

$$=\frac{\pi}{9}\left[\frac{1}{3}x^3\right]_6^9$$

$$=\mathbf{19\pi}$$



2. Find the volume, V, of the solid formed by rotating the region enclosed by the line $y=x+1$, the x-axis, and the line $x=2$ about the x-axis.

[Sol] From the graph,

$$-1\leqslant x\leqslant 2$$

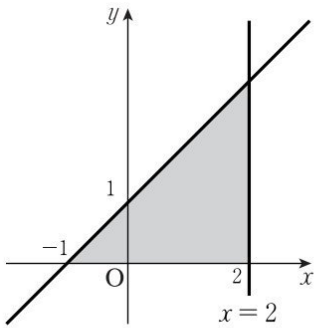
Therefore,

$$\boldsymbol{V}=\pi\int_{-1}^2(x+1)^2dx$$

$$=\pi\int_{-1}^2(x^2+2x+1)dx$$

$$=\pi\left[\frac{1}{3}x^3+x^2+x\right]_{-1}^2$$

$$=\mathbf{9\pi}$$

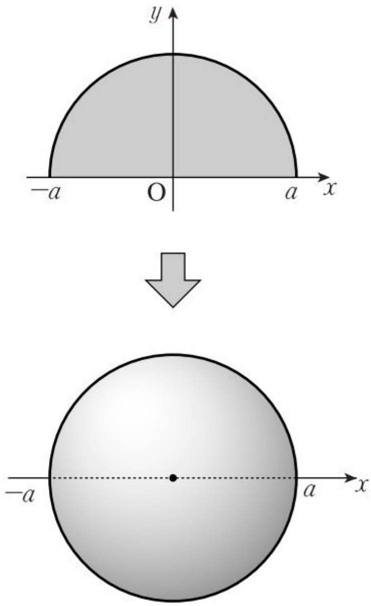


L 166a

KUMON

Volumes

1. A point on a semicircle may be expressed as $y=\sqrt{a^2-x^2}$ (where $-a\leqslant x\leqslant a$). The solid formed by rotating the figure enclosed by this semicircle and the x-axis about the x-axis will be a sphere. Find the volume, V, of the sphere.



[Sol] From the graph,

$$-a\leqslant x\leqslant a$$

From $y^2=\boxed{\boldsymbol{a^2-x^2}}$,

$$V=\pi\int_{-a}^{a}(a^2-x^2)dx$$

$$=2\pi\int_{0}^{a}(a^2-x^2)dx$$

$$=2\pi\left[a^2x-\frac{1}{3}x^3\right]_0^a$$

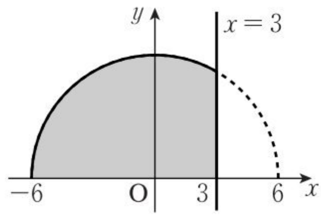
$$=2\pi\left(a^3-\frac{1}{3}a^3\right)$$

$$=\boldsymbol{\frac{4}{3}\pi a^3}$$

Compare this method with the method used on L163a.

L 166b

2. Find the volume, V, of the solid formed by rotating the region enclosed by the semicircle $y=\sqrt{36-x^2}$ (where $-6 \leqslant x \leqslant 6$), the x-axis and the line $x=3$ about the x-axis.



[Sol] From the graph,

$$-6 \leqslant x \leqslant 3$$

From $y^2=36-x^2$,

$$\boldsymbol{V}=\pi\int_{-6}^{3}(36-x^2)dx$$

$$=\pi\left[36x-\frac{1}{3}x^3\right]_{-6}^{3}$$

$$=[(108-9)-(-216+72)]\pi$$

$$\mathbf{=243\pi}$$

L 167a KUMON

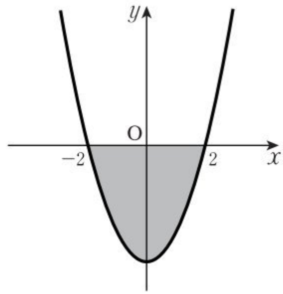
Volumes

1. Find the volume, V, of the solid formed by rotating the region enclosed by $y = x^2 - 4$ and the x-axis about the x-axis.

[Sol] Finding the points of intersection between $y = x^2 - 4$ and the x-axis,

$$x = \pm 2$$

From the graph,



$$-2 \leqq x \leqq 2$$

Therefore,

$$\begin{aligned}
\boldsymbol{V} &= \pi \int_{-2}^{2} (x^2 - 4)^2 dx \\
&= 2\pi \int_{0}^{2} (x^4 - 8x^2 + 16) dx \\
&= 2\pi \left[\frac{1}{5}x^5 - \frac{8}{3}x^3 + 16x \right]_0^2 \\
&= 2\pi \left(\frac{32}{5} - \frac{64}{3} + 32 \right) \\
&\boldsymbol{= \frac{512}{15}\pi}
\end{aligned}$$

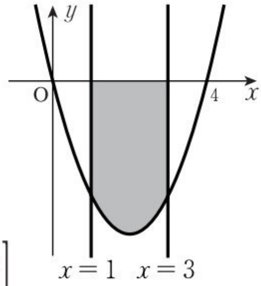
L 167b

2. Find the volume, V, of the solid formed by rotating the region enclosed by $y = x^2 - 4x$, the x-axis, the line $x = 1$ and the line $x = 3$ about the x-axis.

[Sol] From the graph,

$$1 \leqslant x \leqslant 3$$

Therefore,



$$\begin{aligned}
\boldsymbol{V} &= \pi\int_1^3 (x^2 - 4x)^2 dx \\
&= \pi\int_1^3 (x^4 - 8x^3 + 16x^2) dx \\
&= \pi\left[\frac{1}{5}x^5 - 2x^4 + \frac{16}{3}x^3\right]_1^3 \\
&= \pi\left[\left(\frac{243}{5} - 162 + 144\right) - \left(\frac{1}{5} - 2 + \frac{16}{3}\right)\right] \\
&= \boldsymbol{\frac{406}{15}\pi}
\end{aligned}$$

L 168a KUMON

Volumes

1. Find the volume, V, of the solid formed by rotating the region enclosed by $y=3-x^2$ and the line $y=2$ about the x-axis.

[Sol] Finding the x-coordinates of the points of intersection between $y=3-x^2$ and $y=2$,

$$x=\pm 1$$

Therefore,

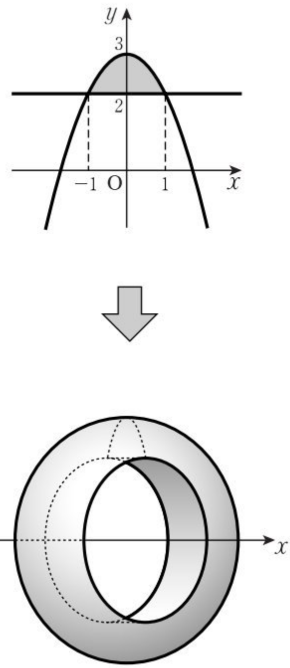
$$V=\pi\int_{-1}^{1}(3-x^2)^2dx-\pi\int_{-1}^{1}2^2dx$$

$$=\pi\int_{-1}^{1}(x^4-6x^2+5)dx$$

$$=2\pi\int_{0}^{1}(x^4-6x^2+5)dx$$

$$=2\pi\left[\frac{1}{5}x^5-2x^3+5x\right]_0^1$$

$$=\mathbf{\frac{32}{5}\pi}$$

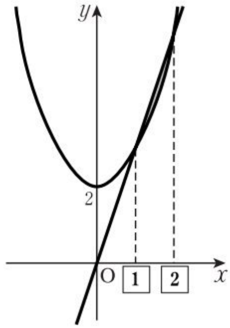


L 168b

2. Find the volume, V, of the solid formed by rotating the region enclosed by $y=x^2+2$ and the line $y=3x$ about the x-axis.

[Sol] Finding the x-coordinates of the points of intersection between $y=x^2+2$ and $y=3x$,

$x=1,\ 2$



Therefore,

$$\begin{aligned}\boldsymbol{V}&=\pi\int_1^2(3x)^2dx-\pi\int_1^2(x^2+2)^2dx\\&=\pi\int_1^2 9x^2dx-\pi\int_1^2(x^4+4x^2+4)dx\\&=\pi\int_1^2(-x^4+5x^2-4)dx\\&=\pi\left[-\frac{1}{5}x^5+\frac{5}{3}x^3-4x\right]_1^2\\&\boldsymbol{=\frac{22}{15}\pi}\end{aligned}$$

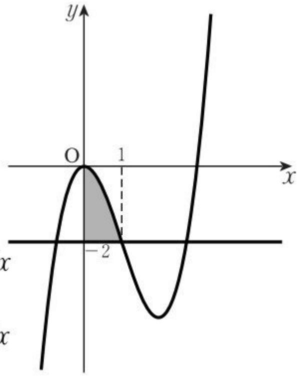
L 169a KUMON

Volumes

1. Find the volume, V, of the solid formed by rotating the shaded region ( ) enclosed by the curve $y=x^3-3x^2$, the line $y=-2$ and the y-axis about the x-axis.

[Sol] Finding the x-coordinates of the points of intersection of $y=x^3-3x^2$ and $y=-2$,

$$x^3-3x^2=-2$$
$$x^3-3x^2+2=0$$
$$(x-1)(x^2-2x-2)=0$$
$$x=1,\ 1\pm\sqrt{3}$$



Therefore,

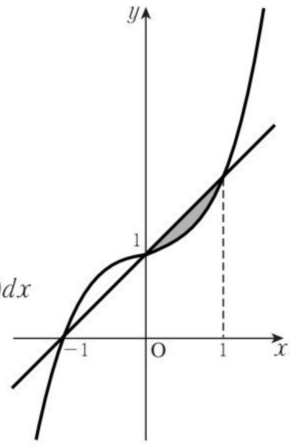
$$\boldsymbol{V}=\pi\int_0^1(-2)^2dx-\pi\int_0^1(x^3-3x^2)^2dx$$
$$=\pi\int_0^1 4dx-\pi\int_0^1(x^6-6x^5+9x^4)dx$$
$$=\pi\int_0^1(-x^6+6x^5-9x^4+4)dx$$
$$=\pi\left[-\frac{1}{7}x^7+x^6-\frac{9}{5}x^5+4x\right]_0^1$$
$$=\pi\left(-\frac{1}{7}+1-\frac{9}{5}+4\right)$$
$$\boldsymbol{=\frac{107}{35}\pi}$$

L 169b

2. Find the volume, V, of the solid formed by rotating the shaded region ( ) enclosed by the curve $y=x^3+1$ and the line $y=x+1$ about the x-axis.

[Sol] Finding the x-coordinates of the points of intersection of $y=x^3+1$ and $y=x+1$,

$$x^3+1=x+1$$
$$x^3-x=0$$
$$x(x^2-1)=0$$
$$x(x+1)(x-1)=0$$
$$x=-1,\ 0,\ 1$$



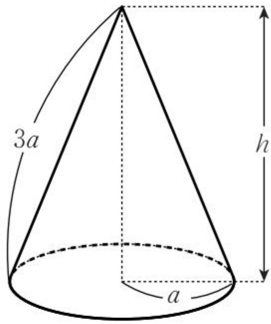
Therefore,

$$\boldsymbol{V}=\pi\int_0^1(x+1)^2dx-\pi\int_0^1(x^3+1)^2dx$$
$$=\pi\int_0^1(x^2+2x+1)dx-\pi\int_0^1(x^6+2x^3+1)dx$$
$$=\pi\int_0^1(-x^6-2x^3+x^2+2x)dx$$
$$=\pi\left[-\frac{1}{7}x^7-\frac{1}{2}x^4+\frac{1}{3}x^3+x^2\right]_0^1$$
$$=\pi\left(-\frac{1}{7}-\frac{1}{2}+\frac{1}{3}+1\right)$$
$$=\boldsymbol{\frac{29}{42}\pi}$$

L 170a

Volumes

1. Given a cone with radius a and slant height $3a$, find the volume, V, by the following steps.



(1) Find the height, h, of the cone.

[Sol] $h^2+a^2=(3a)^2$

$h^2=8a^2$

$\boldsymbol{h=2\sqrt{2}a}$

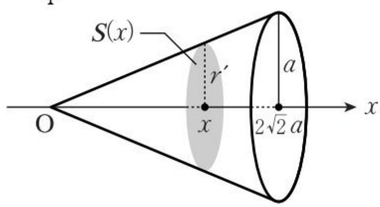
(2) Find the volume, V, of the cone by integration.

[Sol] If the radius of the cross-sectional circle is r',

from $r' : a = x : 2\sqrt{2}a$, $\quad r' = \dfrac{a}{2\sqrt{2}a}x = \dfrac{\sqrt{2}}{4}x$

Therefore,

$$S(x) = \pi\left(\frac{\sqrt{2}}{4}x\right)^2 = \frac{\pi}{8}x^2$$



The range of x-values is $0 \leqq x \leqq 2\sqrt{2}\,a$

Thus,

$$\boldsymbol{V} = \int_0^{2\sqrt{2}a} \frac{\pi}{8}x^2\,dx = \frac{\pi}{8}\left[\frac{1}{3}x^3\right]_0^{2\sqrt{2}a}$$

$$= \frac{\pi}{24}(16\sqrt{2}a^3) \boldsymbol{= \frac{2\sqrt{2}}{3}\pi a^3}$$

☞ This method is called "by integration of a cross-sectional area".

Alternative Solution (by rotation)

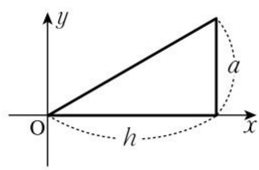
$y = \dfrac{a}{h}x = \dfrac{\sqrt{2}}{4}x$

Using the formula on L165a,

$$\boldsymbol{V} = \pi\int_0^{2\sqrt{2}a}\left(\frac{\sqrt{2}}{4}x\right)^2 dx$$

$$= \pi\left[\frac{1}{24}x^3\right]_0^{2\sqrt{2}a} = \frac{16\sqrt{2}}{24}\pi a^3$$

$$\boldsymbol{= \frac{2\sqrt{2}}{3}\pi a^3}$$

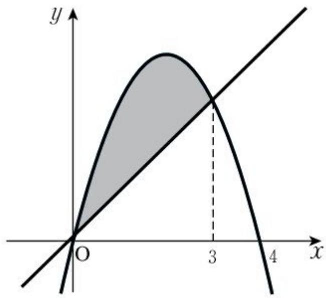


☞ This method is called "by use of the formula for a volume of revolution".

L170b

2. Find the volume, V, of the solid formed by rotating the region enclosed by $y=4x-x^2$ and $y=x$ about the x-axis.

[Sol] Finding the x-coordinates of the points of intersection between $y=4x-x^2$ and $y=x$:



From $4x-x^2=x$,

$$x^2-3x=0$$

$$x=0,\ 3$$

Therefore,

$$\boldsymbol{V}=\pi\int_0^3(4x-x^2)^2dx-\pi\int_0^3x^2dx$$

$$=\pi\int_0^3(x^4-8x^3+15x^2)dx$$

$$=\pi\left[\frac{1}{5}x^5-2x^4+5x^3\right]_0^3$$

$$=\left(\frac{243}{5}-162+135\right)\pi$$

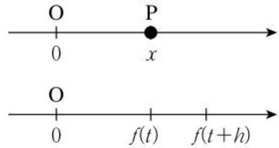
$$\boldsymbol{=\frac{108}{5}\pi}$$

L 171a KUMON

Velocity and Distance

When a point P moves on a number line, then the position x at time t can be expressed as a function $x=f(t)$. The ***average velocity*** of P from time t to time $t+h$ is given by the following formula:

$$\frac{f(t+h)-f(t)}{h}$$



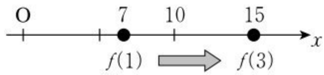
1. Complete the following questions when the position of point P moving on a number line is $x=f(t)=-t^2+8t$. (Assume that the units of measurement are metres for position, seconds for time, and metres/second for velocity.)

(1) Find the position of point P, 1 second after starting, and then again 2 seconds after that. Then, find the average velocity.

[Sol] $x=f(1)=\mathbf{7}$

$x=f(1+2)=f(3)=\mathbf{15}$

$\dfrac{f(3)-f(1)}{2}=\dfrac{8}{2}=\mathbf{4}$



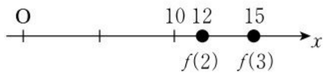
Ans. **4** m/sec

(2) Find the position of point P, 2 seconds after starting, and then again 1 second after that. Then, find the average velocity.

[Sol] $x=f(2)=12$

$x=f(2+1)=f(3)=15$

$\dfrac{f(3)-f(2)}{1}=\dfrac{3}{1}=3$



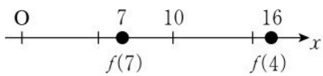
Ans. **3** m/sec

(3) Find the position of point P, 4 seconds after starting, and then again 3 seconds after that. Then, find the average velocity.

[Sol] $x=f(4)=16$

$x=f(4+3)=f(7)=7$

$\dfrac{f(7)-f(4)}{3}=-\dfrac{9}{3}=-3$



Ans. **−3** m/sec

Using the formula for the average velocity from side a, the value of the limit when $h \to 0$, i.e. $\lim_{h \to 0} \dfrac{f(t+h)-f(t)}{h} = f'(t)$, is called the ***velocity*** of point P at time t. If the position of point P is $x = f(t)$, we can express the velocity of point P as:

$$\boxed{v = \frac{dx}{dt} = f'(t)}$$

$\dfrac{dx}{dt}$ denotes "the derivative of x with respect to t".

Ex.

Complete the following questions when the position of point P moving on a number line is $x = f(t) = -t^2 + 8t$. (Units are metres and seconds.)

(1) Express the velocity, v, of point P as a function of time, t.

[Sol] $v = \dfrac{dx}{dt} = (-t^2 + 8t)' = -2t + 8$

(2) Find the velocity of point P, 3 seconds after starting.

[Sol] $v = -2 \times 3 + 8 = 2$

Ans. 2 m/sec

2. Complete the following questions when the position of point P moving on a number line is $x = f(t) = t^3 - 3t^2$. (Units are metres and seconds.)

(1) Express the velocity, v, of point P as a function of time, t.

$v = \dfrac{dx}{dt} = \boldsymbol{3t^2 - 6t}$

Ans. $v = \boldsymbol{3t^2 - 6t}$

(2) Find the velocity of point P, 1 second after starting.

$v = 3 \times 1^2 - 6 \times 1 = \mathbf{-3}$

Ans. **−3** m/sec

(3) Find the velocity of point P, 4 seconds after starting.

$v = 3 \times 4^2 - 6 \times 4 = \mathbf{24}$

Ans. **24** m/sec

When point P moves right on the number line, the velocity has a positive value.
When point P moves left on the number line, the velocity has a negative value.
Velocity can also be called the "rate of change of position".

L 172a KUMON

Velocity and Distance

1. A train running on straight tracks puts on the brake, and then travels a distance of x metres in t seconds, where $x = 28t - 0.56t^2$.

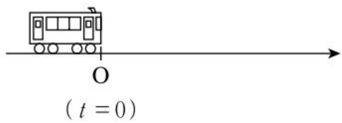
(1) Express the train's velocity, v, after t seconds, as a function of t.

[Sol] $v = \dfrac{dx}{dt} = \mathbf{28 - 1.12}\boldsymbol{t}$

Ans. $v = \mathbf{28 - 1.12}\boldsymbol{t}$

(2) Find the train's velocity at the time it puts on the brake.

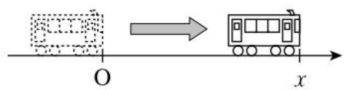
[Sol] When putting on the brake,
the time becomes $t = 0$.
Therefore, $v = 28 - 1.12 \times 0 = 28$



Ans. **28** m/sec

(3) Determine how many seconds it will take for the train to come to a stop after putting on the brake. (The time it takes for the velocity to become $v = 0$.)

[Sol] From $v = 28 - 1.12t = 0$,
$t = 25$



Ans. **25** sec

(4) Find the distance that the train will travel before it comes to a stop, after putting on the brake.

[Sol] $x = 28 \times 25 - 0.56 \times 25^2$
$= 350$

Ans. **350** m

L 172b

2. A train running on straight tracks puts on the brake, and then travels a distance of x metres in t seconds, where $x = 27t - 0.45t^2$.

(1) Find the train's velocity at 10 seconds after putting on the brake.

[Sol] From $v = \dfrac{dx}{dt} = 27 - 0.9t$,

$$v = 27 - 0.9 \times 10 = 18$$

Ans. **18 m/sec**

(2) Find the distance that the train will travel before it comes to a stop, after putting on the brake.

[Sol] From $v = 27 - 0.9t = 0$,

$$t = 30$$

Therefore,

$$x = 27 \times 30 - 0.45 \times 30^2 = 405$$

Ans. **405 m**

3. Another train reaches its maximum velocity 50 seconds after starting. During that time, it travels a distance of x metres, where $x = 0.25t^2$. Find the train's maximum velocity, and the distance travelled before reaching the maximum velocity.

[Sol] Expressing the train's velocity, v, as a derivative with respect to time, t,

From $v = \dfrac{dx}{dt} = 0.5t$,

$$v = 0.5 \times 50 = 25$$

$$x = 0.25 \times 50^2 = 625$$

Ans. Maximum Velocity: **25 m/sec** Distance: **625 m**

L 173a KUMON

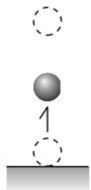
Velocity and Distance

1. A ball thrown upward travels above ground to a height of h metres in t seconds, where $h=24.5t-4.9t^2$.

(1) Express the ball's velocity, v, after t seconds, as a function of t.

[Sol] $v=\dfrac{dh}{dt}=\mathbf{24.5-9.8}\boldsymbol{t}$

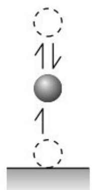
Ans. $v=\mathbf{24.5-9.8}\boldsymbol{t}$



(2) Find the ball's velocity at time 2 seconds.

[Sol] $v=24.5-9.8\times2=4.9$

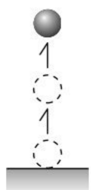
Ans. **4.9** m/sec



(3) Find the ball's velocity at time 4 seconds.

[Sol] $v=24.5-9.8\times4=-14.7$

Ans. **−14.7** m/sec



(4) Determine how many seconds it will take for the ball to reach a maximum height. (The time it takes for the velocity to become $v=0$.)

[Sol] From $v=24.5-9.8t=0$,

$t=2.5$

Ans. **2.5** sec

(5) Determine the time, and the velocity at the moment when the ball falls to the ground again. (The time and velocity at which the height becomes $\boldsymbol{h}=0$.)

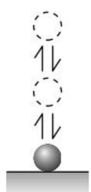
[Sol] From $h=24.5t-4.9t^2=0$,

$t=5$

One solution ($t=0$) corresponds to the time of launch.

Therefore,

$v=24.5-9.8\times5=-24.5$



Ans. Time: **5** sec, Velocity: **−24.5** m/sec

2. A ball thrown upward travels above ground to a height of h metres in t seconds, where $h=29.4t-4.9t^2$.

(1) Find the ball's velocity and height at time 2 seconds.

[Sol] $v=\dfrac{dh}{dt}=29.4-9.8t$

$v=29.4-9.8\times 2=9.8$

$h=29.4\times 2-4.9\times 2^2=39.2$

Ans. Velocity: **9.8 m/sec** Height: **39.2 m**

(2) Find the maximum height that the ball reaches.

[Sol] From $v=29.4-9.8t=0$,

$t=3$

Therefore,

$h=29.4\times 3-4.9\times 3^2=44.1$

Ans. **44.1 m**

(3) Find the ball's velocity at the moment it falls to the ground again.

[Sol] From $h=29.4t-4.9t^2=0$,

$t=6$

One solution ($t=0$) corresponds to the time of launch.

Therefore,

$v=29.4-9.8\times 6=-29.4$

Ans. **$-$29.4 m/sec**

L 174a KUMON

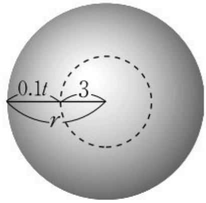
Velocity and Distance

1. A spherical balloon has an original radius of 3 cm that increases at a rate of 0.1 cm per second ($\frac{1}{10}$ cm/sec), after starting to pump in air.

(1) Find the surface area after t seconds.
(Note: The surface area of a sphere with radius r is $4\pi r^2$.)

[Sol] At t seconds, let r be the radius, and let S be the surface area.

$$r = 3 + 0.1t = 3 + \frac{1}{10}t$$



Therefore,

$$S = 4\pi r^2$$
$$= 4\pi\left(3 + \frac{1}{10}t\right)^2$$
$$= \boldsymbol{\frac{\pi}{25}t^2 + \frac{12\pi}{5}t + 36\pi}$$

Ans. $\boldsymbol{\frac{\pi}{25}t^2 + \frac{12\pi}{5}t + 36\pi}$ cm^2

(2) Find the rate of change of the surface area at time 10 seconds.

[Sol] From $v = \dfrac{dS}{dt} = \boxed{\boldsymbol{\frac{2\pi}{25}}}\,t + \boxed{\boldsymbol{\frac{12\pi}{5}}}$,

$$v = \frac{2\pi}{25} \times 10 + \frac{12\pi}{5} = \frac{16\pi}{5}$$

The surface area is growing at a rate of $10.05\ldots$ cm^2 per second (after 10 seconds).

Ans. $\boldsymbol{\frac{16\pi}{5}}$ cm^2/sec

(3) Find after how many seconds the radius reaches 5 cm, and the rate of change of the increasing surface area at that time.

[Sol] From $r = 3 + \frac{1}{10}t = 5$,

$$t = 20$$
$$v = \frac{2\pi}{25} \times 20 + \frac{12\pi}{5} = \frac{8\pi}{5} + \frac{12\pi}{5} = 4\pi$$

The area is now growing at a rate of $12.56\ldots$ cm^2 per second (after 20 seconds).

Ans. At time **20** seconds, $\boldsymbol{4\pi}$ cm^2/sec

2. A spherical balloon with an original radius of 2 cm, has radius, r, which increases at a rate of 2 cm per second. Find the rate of change of the volume when the radius reaches 10 cm.

[Sol] At t seconds, let r be the radius, and let V be the volume.

$$r = \boxed{\mathbf{2+2t}} \quad \cdots①$$

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(\ \mathbf{2+2t}\ )^3 = \frac{\mathbf{32}}{\mathbf{3}}\boldsymbol{\pi t^3 + 32\pi t^2 + 32\pi t} + \frac{\mathbf{32}}{\mathbf{3}}\boldsymbol{\pi}$$

$$v = \frac{dV}{dt} = \boldsymbol{32\pi t^2 + 64\pi t + 32\pi} \quad \cdots②$$

Substituting the radius of 10 cm into ①,

$$10 = 2 + 2t$$

$$t = 4$$

Substituting this value into ② to find the rate of change of the volume,

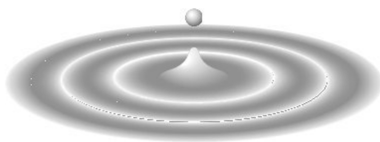
$$v = 32\pi \times 4^2 + 64\pi \times 4 + 32\pi = 800\pi$$

Ans. **800π** cm^3/sec

3. When a rock is thrown into a calm pond, circular waves, which have the same centre, are created. Find the rate of change of the growing surface area which is enclosed by the outermost wave at time 4 seconds, given that its radius grows at 1 m/sec.

[Sol] At t seconds, let r be the radius, and let S be the surface area enclosed by the outermost wave.

$$r = t$$



Therefore,

$$S = \pi r^2 = \pi t^2$$

$$v = \frac{dS}{dt} = 2\pi t$$

When $t = 4$,

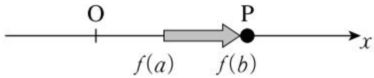
$$v = 2\pi \times 4 = 8\pi$$

Ans. **8π** m^2/sec

L 175a KUMON

Velocity and Distance

Given a point P moving on a number line, if we let the position at time t be $x=f(t)$, the velocity v is:



$$v=\frac{dx}{dt}=f'(t)$$

Therefore, the displacement (the change in position) of point P from $t=a$ to $t=b$ can be found as follows:

$$\boxed{f(b)-f(a)=\int_a^b f'(t)dt=\int_a^b vdt}$$

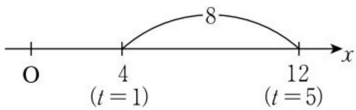
Ex.

A point P moving on a number line with velocity $v=14-4t$, is at $x=4$ when $t=1$. Find its position, x, when $t=5$.

[Sol] The displacement of point P is:

$$f(5)-f(1)=\int_1^5(14-4t)dt$$

$$=\Big[14t-2t^2\Big]_1^5=8$$



Therefore, the position of point P is:

$x=4+8=12$

Ans. $x=12$

1. A point P moving on a number line with velocity $v=20-4t$, is at $x=0$ when $t=5$. Find its position, x, when $t=7$.

[Sol] The displacement of point P is:

$$f(7)-f(5)=\int_5^7(20-4t)dt$$

$$=\Big[20t-2t^2\Big]_5^7=-8$$

A negative displacement means a movement to the left.

Therefore, the position of point P is:

$x=0-8=-8$

Ans. $\boldsymbol{x=-8}$

L 175b

2. A point P moving on a number line with velocity $v = 2t - 6$, is at $x = 8$ when $t = 1$. Find its position, x, when $t = 4$.

[Sol] The displacement of point P is:

$$f(4) - f(1) = \int_1^4 (2t - 6)dt$$
$$= \left[t^2 - 6t\right]_1^4 = -3$$

Therefore, the position of point P is:

$x = 8 - 3 = 5$

Ans. $\boldsymbol{x = 5}$

3. Given that a ball is thrown upward from ground level with a speed of 30 m/sec, its upward velocity after t seconds is expressed as $v = 30 - 10t$ (m/sec).

(1) Find the ball's height at time 2 seconds after being thrown.

[Sol] $h = \int_0^2 v dt = \int_0^2 (30 - 10t)dt = \left[30t - 5t^2\right]_0^2 = 40$

Ans. **40 m**

(2) Find the maximum height that the ball reaches.

[Sol] Finding the time t at which the ball reaches its maximum height:

From $v = 30 - 10t = 0$,

$t = 3$

Therefore, the ball reaches its maximum height after 3 seconds.

$$h = \int_0^3 (30 - 10t)dt = \left[30t - 5t^2\right]_0^3 = 45$$

Ans. **45 m**

L 176a KUMON

Velocity and Distance

Ex.

A point P is moving on a number line, and after t seconds its velocity, v cm/sec, is $v=20-4t$. Find the actual distance travelled by point P from the starting time to time 7 seconds.

[Sol] From $v=20-4t=-4(t-5)$,

$$\begin{cases} \text{When } 0 \leqq t \leqq 5, & v \geqq 0 \\ \text{When } t \geqq 5, & v \leqq 0 \end{cases}$$

Moving to the right ($+$ direction)

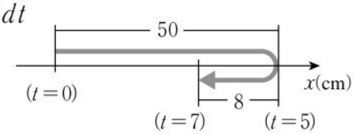
Moving to the left ($-$ direction)

Therefore, letting s be the distance travelled after 7 seconds,

$$s=\int_0^5 (20-4t)dt+\int_5^7 (-20+4t)dt$$

$$=\Big[20t-2t^2\Big]_0^5+\Big[-20t+2t^2\Big]_5^7$$

$$=50+8=58$$



Ans. 58 cm

1. A point P is moving on a number line at a velocity, v cm/sec, where $v=4t-12$. Find the actual distance travelled by point P from the starting time to time 8 seconds.

[Sol] From $v=4t-12=4(t-3)$,

$$\begin{cases} \text{When } 0 \leqq t \leqq 3, & v \leqq 0 \\ \text{When } t \geqq 3, & v \geqq 0 \end{cases}$$

Therefore, letting s be the distance travelled after 8 seconds,

$$s=\int_0^3 (-4t+12)dt+\int_3^8 (4t-12)dt$$

$$=\Big[-2t^2+12t\Big]_0^3+\Big[2t^2-12t\Big]_3^8$$

$$=18+50=68$$

Ans. **68 cm**

L 176b

Generally, given a point P moving at time t with velocity v, the distance, s, travelled from time $t=a$ to $t=b$ is expressed by the following formula:

$$\boxed{s=\int_a^b |v|dt}$$

Note: Distance travelled is always positive (or zero).
Displacements may be positive or negative (or zero).

2. The velocity, v in cm/sec, of a point P is $v=2t-t^2$. Find the distance point P travels from 1 second to 3 seconds after starting.

[Sol] From $v=2t-t^2=-t(t-2)$,

$$|v|=\begin{cases} 2t-t^2 & (\boxed{\mathbf{0}}\leqslant t\leqslant\boxed{\mathbf{2}}) \\ -2t+t^2 & (t\geqslant\boxed{\mathbf{2}}) \end{cases}$$

Therefore, letting s be the distance travelled by point P,

$$s=\int_1^3|v|dt=\int_1^2(2t-t^2)dt+\int_2^3(-2t+t^2)dt$$

$$=\left[t^2-\frac{1}{3}t^3\right]_1^2+\left[-t^2+\frac{1}{3}t^3\right]_2^3$$

$$=\mathbf{2}$$

Ans. **2 cm**

3. Given that a ball is thrown upward from ground level with a speed of 40 m/sec, its upward velocity after t seconds is $v=40-10t$ (m/sec). Find the actual distance the ball travelled 5 seconds after being thrown upward.

[Sol] From $v=40-10t=-10(t-4)$,

$$|v|=\begin{cases} 40-10t & (0\leqslant t\leqslant 4) \\ -40+10t & (t\geqslant 4) \end{cases}$$

Therefore, letting s be the distance travelled by the ball,

$$s=\int_0^5|v|dt=\int_0^4(40-10t)dt+\int_4^5(-40+10t)dt$$

$$=\left[40t-5t^2\right]_0^4+\left[-40t+5t^2\right]_4^5$$

$$=85$$

Ans. **85 m**

L 177a KUMON

Velocity and Distance

1. The velocity of a moving point P on a number line, at time t seconds is $v=12t-3t^2$, and when $t=0$, point P is at the origin.

(1) Express the position, x, of point P at time t, as a function of t. (Refer to the footnote.)

[Sol] $x=\int_0^t v\,dt$

$$=\int_0^t (12t-3t^2)dt=\Big[6t^2-t^3\Big]_0^t=\boldsymbol{6t^2-t^3}$$

Ans. $x=\boldsymbol{6t^2-t^3}$

(2) At what time t does point P return back to the origin?

[Sol] When point P returns to the origin, $x=0$.

From $6t^2-t^3=t^2(6-t)=0$,

$t=6$

Ans. **6 seconds later**

(3) Find the distance, s, that point P travels to return to the origin.

[Sol] From $v=12t-3t^2=-3t(t-4)$,

$$|v|=\begin{cases} 12t-3t^2 & (0\leqq t\leqq 4) \\ -12t+3t^2 & (t\geqq 4) \end{cases}$$

Therefore,

$$s=\int_0^6 |v|dt$$

$$=\int_0^4 (12t-3t^2)dt+\int_4^6 (-12t+3t^2)dt$$

$$=\Big[6t^2-t^3\Big]_0^4+\Big[-6t^2+t^3\Big]_4^6=64$$

Ans. $s=\mathbf{64}$

Note: The variable of integration is t, but t also appears as the upper limit of integration. Thus the following integral remains a function of t, as shown.

$$\int_0^t 2t\,dt=\Big[t^2\Big]_0^t=t^2$$

L 177b

2. The velocity of a moving point P, at time t is $v=|4-t|$ (m/sec).

(1) Letting t be the time in seconds, express the distance travelled, as a function of t, for $0 \leqq t \leqq 4$.

[Sol] $\int_0^t (4-t)dt = \left[4t-\frac{1}{2}t^2\right]_0^t = 4t-\frac{1}{2}t^2$

Ans. $\boldsymbol{4t-\frac{1}{2}t^2}$

(2) Letting t be the time in seconds, express the distance travelled, as a function of t, for $t \geqq 4$.

[Sol] $\int_0^4 (4-t)dt + \int_4^t (-4+t)dt = \left[4t-\frac{1}{2}t^2\right]_0^4 + \left[-4t+\frac{1}{2}t^2\right]_4^t$

$= \frac{1}{2}t^2-4t+16$

Ans. $\boldsymbol{\frac{1}{2}t^2-4t+16}$

(3) Determine how much time is required for point P to travel 26 m.

[Sol] When $0 \leqq t \leqq 4$, substituting $t=4$ into the result of (1),

$4\times4-\frac{1}{2}\times4^2=8$

Thus, the distance covered in this interval of time is too short.
(i.e. We're looking for 26 m; 8 m is too short.)

When $t \geqq 4$, setting the result of (2) equal to 26,

$\frac{1}{2}t^2-4t+16=26$

$t^2-8t-20=0$

$t=-2,\ 10$

Therefore, $t=10$ ($t=-2$ is an extraneous solution)

Ans. **10 seconds**

L 178a KUMON

Velocity and Distance

1. There are two moving points, P and Q. At time t, in seconds, point P has velocity $u=4t-12$ (cm/sec), and point Q has velocity $v=-2t+12$ (cm/sec). Given that points P and Q start from the origin, at the same time, complete the following exercises.

(1) Find the position of point P at time t.

[Sol] Let x_1 be the position of point P, t seconds after starting.

$$x_1=\int_0^t u\,dt$$
$$=\int_0^t (4t-12)dt$$
$$=\Big[2t^2-12t\Big]_0^t=\boldsymbol{2t^2-12t}$$

Ans. $x_1=\boldsymbol{2t^2-12t}$

(2) Find the position of point Q at time t.

[Sol] Let x_2 be the position of point Q, t seconds after starting.

$$x_2=\int_0^t v\,dt$$
$$=\int_0^t (-2t+12)dt$$
$$=\Big[-t^2+12t\Big]_0^t=-t^2+12t$$

Ans. $\boldsymbol{x_2=-t^2+12t}$

(3) Determine at what time (in seconds) points P and Q meet again.

[Sol] From $x_1=x_2$,

$$2t^2-12t=-t^2+12t$$
$$3t^2-24t=0$$

Therefore, $t=8$

One solution ($t=0$) corresponds to starting time.

Ans. **8 seconds later**

L 178b

2. There are two moving points, P and Q. At time t, in seconds, point P has velocity $u=3t^2-6t$ (cm/sec), and point Q has velocity $v=-2t+3$ (cm/sec). Given that points P and Q start from the origin, at the same time, complete the following exercises.

(1) Find at what time (in seconds) points P and Q meet again.

[Sol] Let x_1 be the position of point P, and let x_2 be the position of point Q, t seconds after starting.

$$x_1=\int_0^t u\,dt=\int_0^t(3t^2-6t)dt=\Big[t^3-3t^2\Big]_0^t=t^3-3t^2$$

$$x_2=\int_0^t v\,dt=\int_0^t(-2t+3)dt=\Big[-t^2+3t\Big]_0^t=-t^2+3t$$

From, $x_1=x_2$,

$$t^3-3t^2=-t^2+3t$$

$$t^3-2t^2-3t=0$$

$$t(t-3)(t+1)=0$$

We are looking for the solution $t>0$.

Therefore, $t=3$

Ans. **3 seconds later**

(2) Find the distance that point P moves until meeting point Q.

[Sol] From $u=3t^2-6t=3t(t-2)$,

$$|u|=\begin{cases}-3t^2+6t & (0\leqq t\leqq 2)\\ 3t^2-6t & (t\geqq 2)\end{cases}$$

Therefore, the distance that point P moves is:

$$\int_0^3|u|dt=\int_0^2(-3t^2+6t)dt+\int_2^3(3t^2-6t)dt$$

$$=\Big[-t^3+3t^2\Big]_0^2+\Big[t^3-3t^2\Big]_2^3$$

$$=8$$

Ans. **8 cm**

L 179a KUMON

Velocity and Distance

1. When a plug is pulled out, water starts running out of a water tank. After t minutes, the rate at which water flows out is $v=-t^2+6t$ (litres per minute).

(1) Find the amount of water that flows out in the first t minutes.

[Sol] Let V be the amount of water that flows out in t minutes.

$$V=\int_0^t v\,dt=\int_0^t(-t^2+6t)dt$$

$$=\left[-\frac{1}{3}t^3+3t^2\right]_0^t=\boldsymbol{-\frac{1}{3}t^3+3t^2}$$

Ans. $\boldsymbol{-\frac{1}{3}t^3+3t^2}$ l

(2) Find the amount of water that flows out in the first 2 minutes after pulling out the plug.

[Sol] $V=-\frac{1}{3}\times2^3+3\times2^2=\frac{28}{3}$

Ans. $\boldsymbol{\frac{28}{3}}$ l

(3) Find how much time (in minutes) it will take for all the water to flow out. (i.e. How many minutes later will the velocity become $v=0$?)

[Sol] From $v=-t^2+6t=-t(t-6)=0$,

$t=6$

Ans. **6** minutes later

(4) Find the total amount of water which flows out of the water tank.

[Sol] Since all the water will flow out in 6 minutes,

$$V=-\frac{1}{3}\times6^3+3\times6^2=36$$

Ans. **36** l

L 179b

2. When a plug is pulled out, water starts running out of a water tank. After t minutes, the rate at which water flows out is $v=-2t^2+18t$ (litres per minute).

(1) Find the amount of water that flows out in the first 3 minutes after pulling out the plug.

[Sol] Let V be the amount of water that flows out in t minutes.

$$V=\int_0^t vdt=\int_0^t(-2t^2+18t)dt$$

$$=\left[-\frac{2}{3}t^3+9t^2\right]_0^t=-\frac{2}{3}t^3+9t^2$$

Therefore, determining the amount of water that flows out in 3 minutes,

$$V=-\frac{2}{3}\times3^3+9\times3^2=63$$

Ans. **63 l**

(2) Find the total amount of water which flows out of the water tank.

[Sol] First we find the time, t in minutes, it will take for all the water to flow out.

From $v=-2t^2+18t=-2t(t-9)=0$,

$t=9$

Therefore, since all the water will flow out in 9 minutes,

$$V=-\frac{2}{3}\times9^3+9\times9^2=243$$

Ans. **243 l**

L 180a KUMON

Velocity and Distance

1. A pebble is thrown upward from 24.5 m above ground at a velocity of 19.6 m/sec. The height reached, after t seconds, is expressed as $h = 24.5 + 19.6t - 4.9t^2$ (metres).

(1) Find the maximum height that the pebble reaches.

[Sol] Let v be the pebble's velocity at time t seconds.

From $v = \dfrac{dh}{dt} = 19.6 - 9.8t = 0$,

$t = 2$

Therefore, at time 2 seconds,

$h = 24.5 + 19.6 \times 2 - 4.9 \times 2^2 = 44.1$

Ans. **44.1 m**

Alternative Solution
$h = -4.9t^2 + 19.6t + 24.5$
$\quad = -4.9(t-2)^2 + 44.1$
When $t = 2$, the height h reaches the maximum, **44.1 m**.

(2) Find the instantaneous velocity of the pebble when it hits the ground.

[Sol] From $h = 24.5 + 19.6t - 4.9t^2 = 0$,

$t^2 - 4t - 5 = 0$

$(t-5)(t+1) = 0$

$t = 5$

We are looking for a solution $t > 0$.

Therefore, at time 5 seconds,

$v = 19.6 - 9.8 \times 5$

$\quad = -29.4$

Ans. **-29.4 m/sec**

L 180b

2. A point P is moving on a number line. Point P starts at the origin, and its velocity after t seconds is $v=3t^2-12t$ (cm/sec).

(1) Find the distance point P travels 5 seconds after it starts.

[Sol] From $v=3t^2-12t=3t(t-4)$,

$$|v|=\begin{cases}-3t^2+12t & (0\leqq t\leqq 4)\\ 3t^2-12t & (t\geqq 4)\end{cases}$$

Therefore, letting s be the distance travelled in 5 seconds,

$$s=\int_0^4(-3t^2+12t)dt+\int_4^5(3t^2-12t)dt$$

$$=\Big[-t^3+6t^2\Big]_0^4+\Big[t^3-6t^2\Big]_4^5$$

$$=39$$

Ans. **39 cm**

(2) Find the velocity with which point P returns to the origin.

[Sol] Let x be the position of the point P after t seconds.

From $x=\int_0^t vdt=\int_0^t(3t^2-12t)dt=\Big[t^3-6t^2\Big]_0^t=t^3-6t^2=0$,

$t=6$

Therefore, at time 6 seconds,

$v=3\times6^2-12\times6=36$

Ans. **36 cm/sec**

L 181a KUMON

Summary of Differention and Integration I

1. The curves $y=f(x)=(x+1)^3$ and $y=g(x)=ax^2+bx+c$ (where $a\neq 0$) pass through the same point at $x=0$, and have a common tangent at that point. Also, $y=g(x)$ passes through point $(2,\ 3)$.

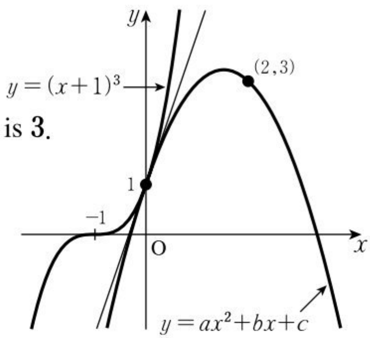
➡L55

(1) Find the common point and the gradient of the common tangent at this point.

[Sol] From $f(0)=1$, the common point is $\mathbf{(0,\ 1)}$.

$$f(x)=x^3+3x^2+3x+1$$
$$f'(x)=3x^2+6x+3=3(x+1)^2$$
$$f'(0)=3$$

Therefore, the gradient of the tangent is $\mathbf{3}$.



(2) Using the given information (that the curves have a common point and a common tangent at this point), find a, b and c.

[Sol] $g'(x)=2ax+b$

Therefore,

$$\begin{cases} g(0)=c=1 & \cdots ① \\ g'(0)=b=3 & \cdots ② \\ g(2)=4a+2b+c=3 & \cdots ③ \end{cases}$$

From ①, ② and ③,

$$\boldsymbol{a=-1,\ b=3,\ c=1}$$

L181b

2. The curves $y=x^3-2x+1$ and $y=x^2+2ax+1$ pass through the same point and have a common tangent at that point.

(1) Find the common point of tangency.

[Sol] Let k be the x-coordinate of the common point of tangency.

Let $f(x)=x^3-2x+1$, and let $g(x)=x^2+2ax+1$

$f'(x)=3x^2-2$, $g'(x)=2x+2a$

At the common point, $f(k)=g(k)$

Therefore, $k^3-2k+1=k^2+2ak+1 \quad \cdots①$

Since the gradients of the common tangent at that point are equal,

$f'(k)=g'(k)$

Therefore, $3k^2-2=2k+2a \quad \cdots②$

From ②, $a=\dfrac{1}{2}(3k^2-2k-2) \quad \cdots③$

Substituting ③ into ①,

$k^3-2k+1=k^2+3k^3-2k^2-2k+1$

$k^2(2k-1)=0$

$k=0,\ \dfrac{1}{2}$

Therefore, the point of tangency is either **(0, 1) or** $\boldsymbol{\left(\dfrac{1}{2},\ \dfrac{1}{8}\right)}$.

(2) Find the value of a.

[Sol] From ③:

If $k=0$, $\boldsymbol{a=-1}$

If $k=\dfrac{1}{2}$, $\boldsymbol{a=-\dfrac{9}{8}}$

L 182a KUMON

Summary of Differention and Integration I

1. Given that the function $f(x)=2x^3+ax^2+bx+c$ has relative extreme values at $x=-2$ and $x=1$, and passes through point $(2,\ 11)$, find the values of a, b and c.

[Sol] $f'(x)=6x^2+2ax+b$

From $f'(-2)=24-4a+b=0$, $\quad 4a-b=24 \quad \cdots①$

From $f'(1)=6+2a+b=0$, $\quad 2a+b=-6 \quad \cdots②$

From $f(2)=16+4a+2b+c=11$, $\quad 4a+2b+c=-5 \quad \cdots③$

From ①+②, $6a=18$

Therefore, $a=3$

Substituting into ②, $b=-12$

Substituting into ③, $c=7$

Therefore, $\boldsymbol{a=3,\ b=-12,\ c=7}$

Alternative Solution

Since there are relative extreme values at $x=-2$ and $x=1$,

Let $f'(x)=m(x+2)(x-1)=m(x^2+x-2)$, (where $m\neq0$)

$$f(x)=\int m(x^2+x-2)dx$$

$$=m\left(\frac{1}{3}x^3+\frac{1}{2}x^2-2x\right)+C \quad \cdots①$$

From $f(2)=\left(\frac{8}{3}+2-4\right)m+C=11$,

$$\frac{2}{3}m+C=11$$

Therefore, $C=11-\frac{2}{3}m$

Substituting into ①,

$$\frac{1}{3}mx^3+\frac{1}{2}mx^2-2mx+\left(11-\frac{2}{3}m\right)=2x^3+ax^2+bx+c$$

Comparing coefficients,

$$\frac{1}{3}m=2,\ \frac{1}{2}m=a,\ -2m=b,\ 11-\frac{2}{3}m=c$$

Therefore, $\boldsymbol{a=3,\ b=-12,\ c=7}$

L 182b

2. The function $f(x)=x^3+ax^2+bx+c$ has relative extreme values at $x=0$ and $x=1$. Find the values of a, b and c for which the product of the relative extreme values of $f(x)$ is a minimum.

➡L61, 62

[Sol] $f'(x)=3x^2+2ax+b$

$f'(0)=b=0 \quad \cdots ①$

$f'(1)=3+2a+b=0 \quad \cdots ②$

From ① and ②,

$$a=-\frac{3}{2},\ b=0$$

Therefore, $f(x)=x^3-\frac{3}{2}x^2+c$

Multiplying the two relative extreme values,

$$\begin{aligned} f(0)\cdot f(1) &= c\left(c-\frac{1}{2}\right) \\ &= c^2-\frac{1}{2}c \\ &= \left(c-\frac{1}{4}\right)^2-\frac{1}{16} \end{aligned}$$

Therefore, when $c=\frac{1}{4}$, the product of the relative extreme values is at a minimum.

Thus, $\boldsymbol{a=-\frac{3}{2},\ b=0,\ c=\frac{1}{4}}$

L 183a KUMON

Summary of Differention and Integration I

1. Given $f(x)=x^3-3x+1$ and $g(x)=x^3+ax^2+b$, find the values of a and b for which the two functions have the same relative maximum and relative minimum values.

➡L76

[Sol] $f'(x)=3x^2-3=3(x+1)(x-1)$

x	$\cdots$	-1	$\cdots$	1	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	↗	relative maximum	↘	relative minimum	↗

Relative Maximum Value:
$f(-1)=3$

Relative Minimum Value:
$f(1)=-1$

$g'(x)=3x^2+2ax=x(3x+2a)$

(i) When $a>0$,

x	$\cdots$	$-\frac{2}{3}a$	$\cdots$	0	$\cdots$
$g'(x)$	$+$	0	$-$	0	$+$
$g(x)$	↗	relative maximum	↘	relative minimum	↗

Relative Maximum Value:
$g\left(-\frac{2}{3}a\right)=\frac{4}{27}a^3+b$

Relative Minimum Value:
$g(0)=b$

Therefore, from $\frac{4}{27}a^3+b=3$ and $b=-1$, $a=3,\ b=-1$

(ii) When $a<0$,

x	$\cdots$	0	$\cdots$	$-\frac{2}{3}a$	$\cdots$
$g'(x)$	$+$	0	$-$	0	$+$
$g(x)$	↗	relative maximum	↘	relative minimum	↗

Relative Maximum Value:
$g(0)=b$

Relative Minimum Value:
$g\left(-\frac{2}{3}a\right)=\frac{4}{27}a^3+b$

Therefore, from $b=3$ and $\frac{4}{27}a^3+b=-1$, $a=-3,\ b=3$

(iii) When $a=0$, $g'(x)=3x^2 \geqslant 0$

Therefore, $y=g(x)$ does not have any relative extreme values.

From (i), (ii) and (iii),

$$\begin{cases} \boldsymbol{a=3} \\ \boldsymbol{b=-1} \end{cases} \qquad \begin{cases} \boldsymbol{a=-3} \\ \boldsymbol{b=3} \end{cases}$$

L183b

2. Given the function $y = ax^2 - 4a^2x + 2a^2 + 8a + 5$ (where $a \neq 0$):

(1) Express the relative minimum value in terms of a.

(2) Let $g(a)$ be the relative minimum value found in (1). Find the value of a that maximizes $g(a)$.

➡K41, L81

[Sol] (1) Let $f(x) = ax^2 - 4a^2x + 2a^2 + 8a + 5$

Since $f(x)$ is a quadratic function, it has a relative minimum value when $a > 0$.

From $f'(x) = 2ax - 4a^2 = 2a(x - 2a)$,

the relative minimum value is:

$$\boldsymbol{f(2a)} = a\cdot(2a)^2 - 4a^2\cdot 2a + 2a^2 + 8a + 5$$
$$= \boldsymbol{-4a^3 + 2a^2 + 8a + 5}$$

Alternative Solution

Completing the square,

$$ax^2 - 4a^2x + 2a^2 + 8a + 5$$
$$= a(x^2 - 4ax + 4a^2) - 4a^3 + 2a^2 + 8a + 5$$
$$= a(x - 2a)^2 - 4a^3 + 2a^2 + 8a + 5$$

Therefore, the relative minimum value is $\boldsymbol{-4a^3 + 2a^2 + 8a + 5}$.

(2) Let $g(a) = -4a^3 + 2a^2 + 8a + 5$

$g'(a) = -12a^2 + 4a + 8 = -4(a-1)(3a+2)$

a	0	$\cdots$	1	$\cdots$
$g'(a)$	$\times$	$+$	0	$-$
$g(a)$	$\times$	$\nearrow$	relative maximum	$\searrow$

Therefore, the value of a that maximizes $g(a)$ is:

$\boldsymbol{a = 1}$

L 184a KUMON

Summary of Differention and Integration I

1. Given the function $f(x)=x^2(3a-x)$, find the maximum and minimum values on the interval $-2\leqq x\leqq 2$. Assume that $0<a<1$.

➡L81, 86

[Sol] $f(x)=x^2(3a-x)$

$f'(x)=-3x(x-2a)$

x	-2	$\cdots$	0	$\cdots$	$2a$	$\cdots$	2
$f'(x)$	$-$	$-$	0	$+$	0	$-$	$-$
$f(x)$	$4(3a+2)$	↘	relative minimum	↗	relative maximum	↘	$4(3a-2)$

$f(-2)=4(3a+2)$, $\quad f(0)=0$

$f(2a)=4a^3$, $\quad f(2)=4(3a-2)$

The maximum value is either $f(-2)$ or $f(2a)$.

Comparing these two values:

$$\begin{aligned} f(2a)-f(-2) &= 4a^3-4(3a+2) \\ &= 4a^3-12a-8 \\ &= 4(a^3-3a-2) \\ &= 4(a+1)^2(a-2)<0 \quad \text{(assuming } 0<a<1) \end{aligned}$$

Therefore, the maximum value is $f(-2)=4(3a+2)$.

The minimum value is either $f(0)$ or $f(2)$.

Comparing these two values:

(i) When $\frac{2}{3}\leqq a<1$, $f(2)\geqq 0$

Since $f(2)\geqq f(0)$, the minimum value is $f(0)=0$.

(ii) When $0<a<\frac{2}{3}$, $f(2)<0$

Since $f(0)>f(2)$, the minimum value is $f(2)=4(3a-2)$.

Ans: **Maximum value: $f(-2)=4(3a+2)$**

Minimum value: $\begin{cases} f(2)=4(3a-2) & \left(\text{when } 0<a<\frac{2}{3}\right) \\ f(0)=0 & \left(\text{when } \frac{2}{3}\leqq a<1\right) \end{cases}$

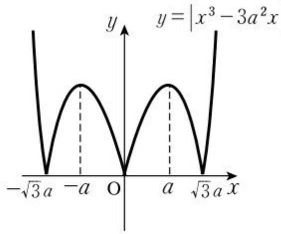
L 184b

2. Find the maximum value of the function $y=\left|\, x^3-3a^2x \,\right|$ (where $a>0$), on the interval $0\leqslant x\leqslant 1$.

➡L84

[Sol] Let $f(x)=x^3-3a^2x$

$$f'(x)=3x^2-3a^2=3(x+a)(x-a)$$



Let $f(x)=0$ to find x:

From $x^3-3a^2x=x(x+\sqrt{3}a)(x-\sqrt{3}a)=0$,

$$x=0,\ -\sqrt{3}a,\ \sqrt{3}a$$

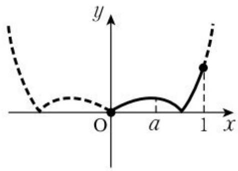
Therefore,

$$|f(x)|=\begin{cases} x^3-3a^2x & (\text{when } x\geqslant\sqrt{3}a) \\ -x^3+3a^2x & (\text{when } 0\leqslant x<\sqrt{3}a) \end{cases}$$

$$|f(a)|=|a^3-3a^3|=|-2a^3|=2a^3$$

$$|f(1)|=\begin{cases} 1-3a^2 & (\text{when } 0<a\leqslant\dfrac{1}{\sqrt{3}}) \\ 3a^2-1 & (\text{when } a>\dfrac{1}{\sqrt{3}}) \end{cases}$$

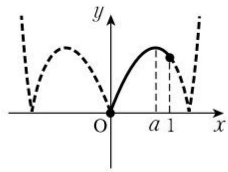
(i)



The maximum is either at $x=a$ or $x=1$.

Find the value of a for which $|f(a)|=|f(1)|$:

(ii)

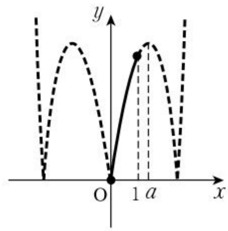


When $0<a\leqslant\dfrac{1}{\sqrt{3}}$,

$$2a^3=1-3a^2$$

$$(a+1)^2(2a-1)=0, \quad \text{so } a=\frac{1}{2}$$

(iii)



When $a>\dfrac{1}{\sqrt{3}}$,

$$2a^3=3a^2-1$$

$$(a-1)^2(2a+1)=0, \quad \text{so } a=1$$

Therefore:

(i) **When $0<a<\frac{1}{2}$,** the maximum value is **$1-3a^2$, at $x=1$**.

(ii) **When $\frac{1}{2}\leqslant a\leqslant 1$,** the maximum value is **$2a^3$, at $x=a$**.

(iii) **When $a>1$,** the maximum value is **$3a^2-1$, at $x=1$**.

L 185a KUMON

Summary of Differention and Integration I

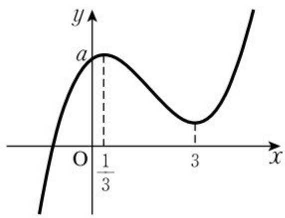
1. Given the function $f(x)=x^3-5x^2+3x+a$, find the conditions for which $f(x)>0$ is always true for any positive value of x.

➡L109

[Sol] $f'(x)=3x^2-10x+3$

$=(3x-1)(x-3)$

x	0	$\cdots$	$\frac{1}{3}$	$\cdots$	3	$\cdots$
$f'(x)$	$+$	$+$	0	$-$	0	$+$
$f(x)$	a	↗	relative maximum	↘	relative minimum	↗



From the table, we can see that $f(x)>0$ for all $x>0$ under the following conditions:

$$\begin{cases} f(0)=a \geqq 0 & \cdots ① \\ f(3)=a-9>0 & \cdots ② \end{cases}$$

From ① and ②,

$\boldsymbol{a>9}$

L 185b

2. Find the range of values of a for which the equation $x^3-3a^2x+2=0$ has 3 different real roots.

➡L105, 106

[Sol]

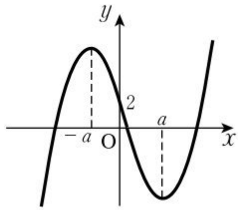
Let $f(x)=x^3-3a^2x+2$

$f'(x)=3x^2-3a^2=3(x+a)(x-a)$

(i) When $a=0$,

$x^3+2=0$ does not have 3 different real number roots.

(ii) When $a>0$,

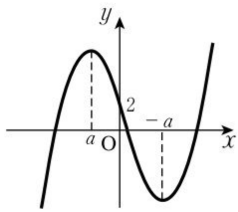


$f(x)$ has a relative minimum at $x=a$, and a relative maximum at $x=-a$.

As $f(0)=2>0$, there are 3 different roots only if $f(a)=-2a^3+2<0$.

Therefore, $2a^3>2$, i.e. $a>1$.

(iii) When $a<0$,



$f(x)$ has a relative minimum at $x=-a$, and a relative maximum at $x=a$.

As $f(0)=2>0$, there are 3 different roots only if $f(-a)=2a^3+2<0$.

Therefore, $2a^3<-2$, i.e. $a<-1$.

From (i), (ii) and (iii), $\boldsymbol{a<-1,\ a>1}$.

Alternative Solution

For either case ($a>0$ or $a<0$),

there are 3 different real number roots when $f(a)\cdot f(-a)<0$.

Therefore,

$$(-2a^3+2)(2a^3+2)=-4(a-1)(a^2+a+1)(a+1)(a^2-a+1)<0$$

Since a^2+a+1 and a^2-a+1 are always positive,

$$(a-1)(a+1)>0$$

Therefore, $\boldsymbol{a<-1,\ a>1}$.

L 186a KUMON

Summary of Differention and Integration I

1. The function$f(x)=a(x-2)(x-3)(x-b)$, where $f(2)=f(3)=0$, has a relative maximum value of 4 at $x=1$.

(1) Find the value of b.

(2) Find the value of x where the function has a relative minimum value.

[Sol] $f(x)=a[x^3-(5+b)x^2+(6+5b)x-6b]$

$f'(x)=a[3x^2-2(5+b)x+(6+5b)]$ $\cdots$①

(1) From $f'(1)=0$,

substituting $x=1$ into ①,

$$3-2(5+b)+(6+5b)=3b-1=0$$

Therefore, $\boldsymbol{b=\frac{1}{3}}$

(2) From $f(1)=4$,

substituting $b=\frac{1}{3}$ and $x=1$ into $f(x)$,

$$a(1-2)(1-3)\left(1-\frac{1}{3}\right)=\frac{4}{3}a=4$$

Therefore, $a=3$

Substituting $a=3$ and $b=\frac{1}{3}$ into ①,

$$f'(x)=3\left(3x^2-\frac{32}{3}x+\frac{23}{3}\right)$$
$$=9x^2-32x+23=(x-1)(9x-23)$$

x	$\cdots$	1	$\cdots$	$\frac{23}{9}$	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	relative maximum	$\searrow$	relative minimum	$\nearrow$

Thus, $f(x)$ has a relative minimum value at $\boldsymbol{x=\frac{23}{9}}$.

L186b

2. Find the cubic function $f(x)$ that satisfies the following conditions:

(i) When it is divided by $(x-2)^2$, the remainder is $2x+1$.

(ii) It has a relative extreme value of 2, at $x=1$.

[Sol] For condition (i), let the quotient be $ax+b$.

Expand $(x-2)^2$ first, before differentiating. Use the formula on L49b.

$$f(x)=(x-2)^2(ax+b)+2x+1$$

$$\begin{aligned} f'(x) &= (2x-4)(ax+b)+a(x-2)^2+2 \\ &= 2ax^2+(2b-4a)x-4b+ax^2-4ax+4a+2 \\ &= 3ax^2+(2b-8a)x+4a-4b+2 \end{aligned}$$

Using condition (ii),

$$\begin{aligned} f'(1) &= 3a+2b-8a+4a-4b+2 \\ &= -a-2b+2=0 \quad \cdots① \end{aligned}$$

$$f(1)=a+b+3=2 \quad \cdots②$$

From ① and ②, $a=-4$, $b=3$

Therefore,

$$\begin{aligned} \boldsymbol{f(x)} &= (x-2)^2(-4x+3)+2x+1 \\ &= (x^2-4x+4)(-4x+3)+2x+1 \\ &\boldsymbol{= -4x^3+19x^2-26x+13} \end{aligned}$$

Alternative method of differentiating

Expand first, before differentiating.

$$\begin{aligned} f(x) &= (x-2)^2(ax+b)+2x+1 \\ &= (x^2-4x+4)(ax+b)+2x+1 \\ &= ax^3+(b-4a)x^2+(4a-4b+2)x+4b+1 \end{aligned}$$

$$f'(x)=3ax^2+(2b-8a)x+4a-4b+2$$

L 187a KUMON

Summary of Differention and Integration I

1. Given that line $y=ax+b$ passes through point $(1,\ 2)$, find the values of a and b that give the minimum value of the definite integral $\int_{-1}^{1}(ax+b)^2dx$.

➡L 138

[Sol] Since line $y=ax+b$ passes through $(1,\ 2)$,

from $2=a+b$, $\quad b=2-a \qquad \cdots①$

$$\int_{-1}^{1}(ax+b)^2dx=\int_{-1}^{1}(a^2x^2+2abx+b^2)dx$$
$$=2\int_{0}^{1}(a^2x^2+b^2)dx$$
$$=2\left[\frac{a^2}{3}x^3+b^2x\right]_0^1$$
$$=2\left(\frac{a^2}{3}+b^2\right) \qquad \cdots②$$

Substituting ① into ②,

$$2\left(\frac{a^2}{3}+b^2\right)=2\left[\frac{a^2}{3}+(2-a)^2\right]$$
$$=\frac{8}{3}(a^2-3a+3)$$
$$=\frac{8}{3}\left[\left(a-\frac{3}{2}\right)^2+\frac{3}{4}\right]$$

Therefore, when $a=\frac{3}{2}$, the definite integral has a minimum value.

At that value, from ①, $b=2-a=2-\frac{3}{2}=\frac{1}{2}$

Thus, $\boldsymbol{a=\frac{3}{2},\ b=\frac{1}{2}}$

L187b

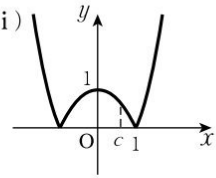
2. Find the value of c that satisfies the integral equation $\int_0^c |x^2-1|dx = c$. Assume that $c>0$.

➡L127

[Sol] (i) When $0<c<1$,

$$\int_0^c |x^2-1|dx = \int_0^c (-x^2+1)dx$$
$$= \left[-\frac{1}{3}x^3+x\right]_0^c$$
$$= -\frac{c^3}{3}+c = c$$

(i)



Thus, $-\dfrac{c^3}{3}=0$, so $c=0$.

Since $0<c<1$, $c=0$ does not satisfy this condition.

(ii) When $c \geqq 1$,

$$\int_0^c |x^2-1|dx = \int_0^1 (-x^2+1)dx + \int_1^c (x^2-1)dx$$
$$= \left[-\frac{1}{3}x^3+x\right]_0^1 + \left[\frac{1}{3}x^3-x\right]_1^c$$
$$= \frac{2}{3}+\frac{c^3}{3}-c-\frac{1}{3}+1$$
$$= \frac{c^3}{3}-c+\frac{4}{3}=c$$

Thus, $\dfrac{c^3}{3}-2c+\dfrac{4}{3}=0$,

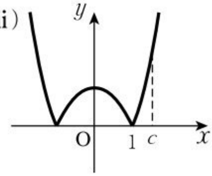
$c^3-6c+4=0$

$(c-2)(c^2+2c-2)=0$

$c=2,\ -1\pm\sqrt{3}$

Since $c \geqq 1$, $c=2$

(ii)



From (i) and (ii), $\boldsymbol{c=2}$

L 188a KUMON

Summary of Differention and Integration I

1. Determine the functions $f(x)$ and $g(x)$ (both functions of x) that satisfy the following integral relationships.

$$f(x)=1+\int_0^x g(t)dt\text{, and } g(x)=x(x-1)+\int_{-1}^1 f(t)dt$$

➡L131, 136

[Sol] Let $c=\int_{-1}^1 f(t)dt$ $\cdots$①

This integral is between fixed limits $[-1,\ 1]$ and will give a constant result, i.e. it is not a function of x.

$$g(x)=x(x-1)+c$$

Therefore,

$$\begin{aligned}f(x)&=1+\int_0^x[t(t-1)+c]dt\\&=1+\int_0^x(t^2-t+c)dt\\&=1+\left[\frac{1}{3}t^3-\frac{1}{2}t^2+ct\right]_0^x\\&=\frac{1}{3}x^3-\frac{1}{2}x^2+cx+1\qquad\cdots②\end{aligned}$$

Substituting ② into ①,

$$\begin{aligned}c&=\int_{-1}^1 f(t)dt\\&=\int_{-1}^1\left(\frac{1}{3}t^3-\frac{1}{2}t^2+ct+1\right)dt\\&=2\int_0^1\left(-\frac{1}{2}t^2+1\right)dt\\&=2\left[-\frac{1}{6}t^3+t\right]_0^1=\frac{5}{3}\end{aligned}$$

Thus, $c=\frac{5}{3}$

Therefore,

$$\boldsymbol{f(x)=\frac{1}{3}x^3-\frac{1}{2}x^2+\frac{5}{3}x+1}$$

$$\boldsymbol{g(x)=x^2-x+\frac{5}{3}}$$

L 188b

2. Given the cubic function $f(x)=x^3-3x+\dfrac{8}{3}\displaystyle\int_0^1 f(x)dx$, complete the following exercises.

➡L136, 63

(1) Find the derivative, $f'(x)$.

[Sol] Let $a=\displaystyle\int_0^1 f(x)dx$ $\cdots$①

Then $f(x)=x^3-3x+\dfrac{8}{3}a$ $\cdots$②

Taking the derivative of ②,

$\boldsymbol{f'(x)=3x^2-3}$

(2) Determine the function $f(x)$.

[Sol] Substituting ② into ①,

$$a=\int_0^1\left(x^3-3x+\frac{8}{3}a\right)dx$$
$$=\left[\frac{1}{4}x^4-\frac{3}{2}x^2+\frac{8}{3}ax\right]_0^1$$
$$=\frac{1}{4}-\frac{3}{2}+\frac{8}{3}a$$

Therefore, $a=\dfrac{3}{4}$

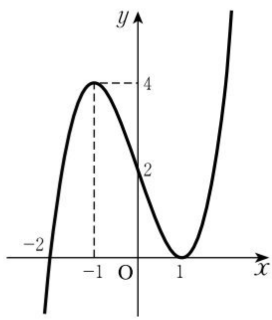
Thus, $\boldsymbol{f(x)=x^3-3x+2}$

(3) Sketch the function $y=f(x)$.

[Sol] $f(x)=x^3-3x+2=(x+2)(x-1)^2$

$f'(x)=3x^2-3=3(x+1)(x-1)$

x	$\cdots$	-1	$\cdots$	1	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	↗	4	↘	0	↗



L 189a KUMON

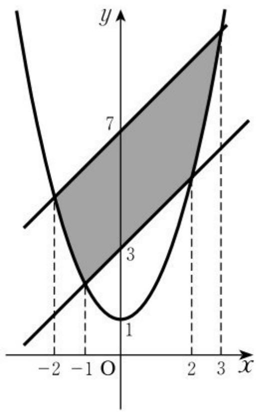
Summary of Differention and Integration I

1. Find the area, S, enclosed by $y=x^2+1$, $y=x+3$ and $y=x+7$.

➡L148

[Sol] Let S_1 be the area enclosed by $y=x^2+1$ and $y=x+7$. Finding the x-coordinates of the points of intersection of $y=x^2+1$ and $y=x+7$,

$$x^2+1=x+7$$
$$x^2-x-6=(x-3)(x+2)=0$$
$$x=-2,\ 3$$



Therefore,

$$S_1=\int_{-2}^{3}[(x+7)-(x^2+1)]dx$$
$$=-\int_{-2}^{3}(x^2-x-6)dx$$
$$=-\int_{-2}^{3}(x+2)(x-3)dx$$
$$=\frac{1}{6}(3+2)^3=\frac{125}{6}$$

Let S_2 be the area enclosed by $y=x^2+1$ and $y=x+3$. Finding the x-coordinates of the points of intersection of $y=x^2+1$ and $y=x+3$,

$$x^2+1=x+3$$
$$x^2-x-2=(x+1)(x-2)=0$$
$$x=-1,\ 2$$

Therefore,

$$S_2=\int_{-1}^{2}[(x+3)-(x^2+1)]dx$$
$$=-\int_{-1}^{2}(x^2-x-2)dx$$
$$=-\int_{-1}^{2}(x+1)(x-2)dx$$
$$=\frac{1}{6}(2+1)^3=\frac{9}{2}$$

Therefore, the area, S, enclosed by the curve and the lines is:

$$S=S_1-S_2=\frac{125}{6}-\frac{9}{2}=\mathbf{\frac{49}{3}}$$

L189b

2. Find the values of a and b that give a minimum value of the area, S, for the region enclosed by the curve $y=-x^2+ax+b$ (which passes through point $(1,\ 2)$) and the curve $y=\frac{1}{2}x^2$.

➡L159

[Sol] Since $y=-x^2+ax+b$ passes through point $(1,\ 2)$,

from $2=-1+a+b$,

$b=3-a$ ⋯①

Finding the x-coordinates of the points of intersection between $y=-x^2+ax+b$ and $y=\frac{1}{2}x^2$:

From $-x^2+ax+b=\frac{1}{2}x^2$,

$x^2-\frac{2}{3}ax-\frac{2}{3}b=0$

Let the solutions of $x^2-\frac{2}{3}ax-\frac{2}{3}b=0$ be α and β (where $\alpha<\beta$),

$\alpha+\beta=\frac{2}{3}a,\quad \alpha\beta=-\frac{2}{3}b$

Therefore,

$$\begin{aligned}
S&=\int_\alpha^\beta\left[(-x^2+ax+b)-\frac{1}{2}x^2\right]dx\\
&=-\frac{3}{2}\int_\alpha^\beta\left(x^2-\frac{2}{3}ax-\frac{2}{3}b\right)dx\\
&=\frac{1}{4}(\beta-\alpha)^3\\
&=\frac{1}{4}\left[\sqrt{(\alpha+\beta)^2-4\alpha\beta}\,\right]^3\\
&=\frac{1}{4}\left(\sqrt{\frac{4}{9}a^2+\frac{8}{3}b}\,\right)^3\\
&=\frac{1}{4}\left[\sqrt{\frac{4}{9}a^2+\frac{8}{3}(3-a)}\,\right]^3\\
&=\frac{2}{27}\left[\sqrt{(a-3)^2+9}\,\right]^3
\end{aligned}$$

Use the formula on L124b.

Substituting ①.

Therefore, when $a=3$, the area, S, will be at its minimum value.

At that value, from ①, $b=0$.

Therefore the area, S, will be at its minimum value when $\boldsymbol{a=3}$ and $\boldsymbol{b=0}$.

L 190a KUMON

Summary of Differention and Integration I

1. Given that the curve $y=-x^3+2x^2$ has a tangent (at a point other than the origin) that passes through the origin, complete the following exercises.

(1) Find the equation of the tangent.

(2) Find the area of the region enclosed by the tangent and the curve.

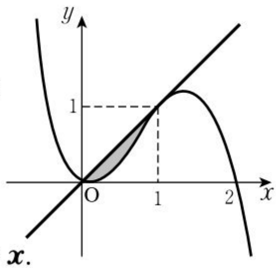
[Sol] (1) Let a (where $a \neq 0$) be the x-coordinate of a point of tangency.

$$y' = -3x^2+4x$$

When $x=a$, the slope of the tangent is $-3a^2+4a$.

Since the tangent passes through the origin, the equation of the tangent is

$$y=(-3a^2+4a)x \quad \cdots ①$$



The point of tangency is $(a, -a^3+2a^2)$, so

$$-a^3+2a^2=(-3a^2+4a)a$$
$$2a^3-2a^2=2a^2(a-1)=0$$

Since $a \neq 0$, $a=1$

From ①, the equation of the tangent is $\boldsymbol{y=x}$.

> Alternative Solution
> Since the tangent passes through the origin, let the equation of the tangent be $y=mx$.
> $$-x^3+2x^2=mx$$
> $$-x^3+2x^2-mx=-x(x^2-2x+m)=0$$
> $x=0$ corresponds to the tangent passing through the origin. The other solution should correspond to a point of tangency, so $x^2-2x+m=0$ must have a repeated solution.
> From $\dfrac{D}{4}=1-m=0$, $m=1$.
> Thus, the equation of the tangent is $\boldsymbol{y=x}$.

(2) Integrating between $x=0$ and $x=1$,

$$\int_0^1 [x-(-x^3+2x^2)]dx = \left[\frac{1}{4}x^4-\frac{2}{3}x^3+\frac{1}{2}x^2\right]_0^1$$
$$=\frac{1}{4}-\frac{2}{3}+\frac{1}{2}=\frac{1}{12}$$

Therefore, the area of the enclosed region is $\boldsymbol{\dfrac{1}{12}}$.

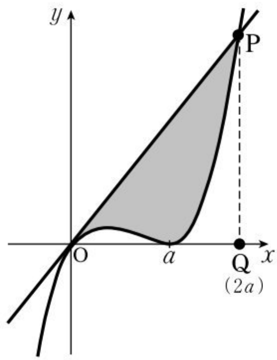
L 190b

2. Curve $y = x(x-a)^2$ (where $a > 1$) has a tangent, l, at the origin, O. Line l also intersects the curve at another point, P, other than O. The perpendicular from P to the x-axis meets the x-axis at Q. Let S_1 be the area of triangle OPQ, and let S_2 be the region enclosed by the curve $y = x(x-a)^2$ and the tangent l.

(1) Find S_1 and S_2 in terms of a.

[Sol] Let $f(x) = x(x-a)^2 = x^3 - 2ax^2 + a^2x$

$f'(x) = 3x^2 - 4ax + a^2$



As $f'(0) = a^2$, the tangent l is:

$y = a^2x$

Finding the coordinates of P,

from $x^3 - 2ax^2 + a^2x = a^2x$,

$x^2(x-2a) = 0$

$x = 0,\ 2a$

Therefore,

$\mathrm{P}(2a,\ 2a^3)$, $\mathrm{Q}(2a,\ 0)$

Therefore,

$$\boldsymbol{S_1} = \frac{1}{2}\cdot \mathrm{OQ}\cdot \mathrm{PQ} = \frac{1}{2}\cdot 2a\cdot 2a^3 = \boldsymbol{2a^4}$$

$$\boldsymbol{S_2} = \int_0^{2a}[a^2x - (x^3 - 2ax^2 + a^2x)]dx$$

$$= \int_0^{2a}(-x^3 + 2ax^2)dx$$

$$= \left[-\frac{1}{4}x^4 + \frac{2}{3}ax^3\right]_0^{2a}$$

$$= \boldsymbol{\frac{4}{3}a^4}$$

(2) Find the ratio of the areas $S_1 : S_2$.

[Sol] $S_1 : S_2 = 2a^4 : \frac{4}{3}a^4 = \mathbf{3 : 2}$

L 191 a

Summary of Differention and Integration II

1. Given the function $f(x)=x^3-3x^2+2x$:

(1) Find the equation of the line that is tangent to curve $y=f(x)$ at point $(a,\ f(a))$.

(2) Find the range of values of p for which the curve has 3 different tangents passing through point $(0,\ p)$.

➡L58

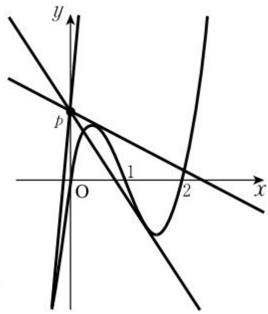
[Sol] (1) $f'(x)=3x^2-6x+2$

$f'(a)=3a^2-6a+2$

The equation of the tangent is:

$\boldsymbol{y}=(3a^2-6a+2)(x-a)+a^3-3a^2+2a$

$\boldsymbol{=(3a^2-6a+2)x-2a^3+3a^2}$ $\cdots$①



(2) Substituting the coordinates $(0,\ p)$ into ①,

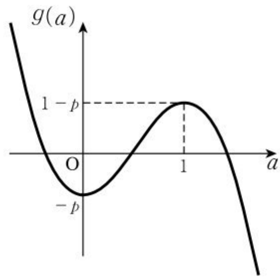
$-2a^3+3a^2=p$

Since 3 tangent lines pass through that point, then the equation $-2a^3+3a^2-p=0$ must have 3 roots for a.

Let $g(a)=-2a^3+3a^2-p$

$g'(a)=-6a^2+6a=-6a(a-1)$

a	$\cdots$	0	$\cdots$	1	$\cdots$
$g'(a)$	$-$	0	$+$	0	$-$
$g(a)$	↘	relative minimum	↗	relative maximum	↘



$g(a)=0$ has 3 roots,

when the relative minimum value <0

and the relative maximum value >0

From $g(0)=-p<0$ and $g(1)=1-p>0$,

$\boldsymbol{0<p<1}$

L 191b

2. Find the range of values of k for which $x^3-3x^2 \geqslant k(3x^2-12x-4)$ is true for all values $x \geqslant 0$. Assume that $k>0$.

➡L77, 109

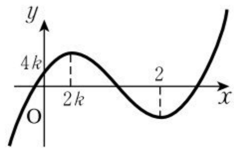
[Sol] Let $f(x)=x^3-3x^2-k(3x^2-12x-4)$

$$f(x)=x^3-3(1+k)x^2+12kx+4k$$

Now $f(0)=4k>0$ as $k>0$.

Also $f'(x)=3x^2-6(1+k)x+12k$

$$=3(x-2)(x-2k)$$



(i) When $0<2k<2$, i.e. when $0<k<1$,

there is a relative minimum at $x=2$.

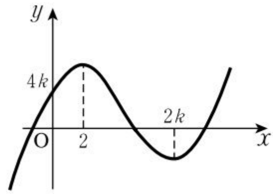
From $f(2)=8-3(1+k)\cdot 4+24k+4k=16k-4$,

we require $f(2) \geqslant 0$, so $16k-4 \geqslant 0$, i.e. $k \geqslant \dfrac{1}{4}$

Therefore, $\dfrac{1}{4} \leqslant k<1$

(ii) When $2k=2$, i.e. when $k=1$,

$f'(x)=3(x-2)^2 \geqslant 0$, so $f(x)$ has no relative extreme values and is always increasing. As $f(0)=4$, $f(x)>0$ for all $x \geqslant 0$.



(iii) When $2k>2$, i.e. when $k>1$,

there is a relative minimum at $x=2k$.

From $f(2k)=8k^3-3(1+k)\cdot 4k^2+24k^2+4k$

$$=-4k(k^2-3k-1)$$

$$=-4k\left[k-\left(\frac{3-\sqrt{13}}{2}\right)\right]\left[k-\left(\frac{3+\sqrt{13}}{2}\right)\right],$$

we require $f(2k) \geqslant 0$ and $k>1$, so

$$1<k \leqslant \frac{3+\sqrt{13}}{2}$$

From (i), (ii) and (iii), $\boldsymbol{\dfrac{1}{4} \leqslant k \leqslant \dfrac{3+\sqrt{13}}{2}}$

L 192a KUMON

Summary of Differention and Integration II

1. Find the range of values of h for which the function $f(x)=x+h$ satisfies the integral inequality: $\int_0^1 [f(x)]^2 dx \leqslant \int_0^1 xf(x)dx$.

[Sol] $\int_0^1 [f(x)]^2 dx = \int_0^1 (x+h)^2 dx = \int_0^1 (x^2+2hx+h^2)dx$

$$= \left[\frac{1}{3}x^3 + hx^2 + h^2 x\right]_0^1$$

$$= \frac{1}{3} + h + h^2$$

$$\int_0^1 xf(x)dx = \int_0^1 x(x+h)dx = \int_0^1 (x^2+hx)dx$$

$$= \left[\frac{1}{3}x^3 + \frac{h}{2}x^2\right]_0^1$$

$$= \frac{1}{3} + \frac{h}{2}$$

From $\frac{1}{3}+h+h^2 \leqslant \frac{1}{3}+\frac{h}{2}$,

$$h^2 + \frac{h}{2} \leqslant 0$$

$$h\left(h+\frac{1}{2}\right) \leqslant 0$$

Therefore, $\boldsymbol{-\frac{1}{2} \leqslant h \leqslant 0}$

L 192b

2. Suppose that $f(x)$ is a quadratic function in x and $\int_{-1}^{1} x^2 f(x)dx = 2$. Find the function $f(x)$ that gives the minimum value of $\int_{-1}^{1} [f(x)]^2 dx$.

➡L138

[Sol] Let $f(x) = ax^2 + bx + c$ (where $a \neq 0$)

$$\int_{-1}^{1} x^2 f(x)dx = \int_{-1}^{1} x^2(ax^2+bx+c)dx = \int_{-1}^{1}(ax^4+bx^3+cx^2)dx$$

$$= 2\int_{0}^{1}(ax^4+cx^2)dx = 2\left[\frac{a}{5}x^5 + \frac{c}{3}x^3\right]_0^1$$

$$= 2\left(\frac{a}{5}+\frac{c}{3}\right) = 2$$

Therefore, $c = 3 - \frac{3}{5}a \quad \cdots ①$

$$\int_{-1}^{1}[f(x)]^2 dx = \int_{-1}^{1}[a^2x^4 + 2abx^3 + (2ca+b^2)x^2 + 2bcx + c^2]dx$$

$$= 2\int_{0}^{1}[a^2x^4 + (2ca+b^2)x^2 + c^2]dx$$

$$= 2\left[\frac{a^2}{5}x^5 + \frac{2ca+b^2}{3}x^3 + c^2x\right]_0^1$$

$$= 2\left(\frac{a^2}{5} + \frac{2ca+b^2}{3} + c^2\right) \quad \cdots ②$$

Substituting ① into ②,

$$2\left[\frac{a^2}{5} + \frac{2}{3}a\left(3-\frac{3}{5}a\right) + \frac{b^2}{3} + \left(3-\frac{3}{5}a\right)^2\right]$$

$$= 2\left(\frac{4}{25}a^2 - \frac{8}{5}a + \frac{b^2}{3} + 9\right)$$

$$= \frac{8}{25}(a-5)^2 + \frac{2}{3}b^2 + 10$$

Therefore, when $a = 5$ and $b = 0$ this integral will be at a minimum value.

At that value, from ①, $c = 3 - \frac{3}{5}a = 0$

Thus, $\boldsymbol{f(x) = 5x^2}$

L 193a KUMON

Summary of Differention and Integration II

1. Determine the functions $f(x)$ and $g(x)$ (both functions of x, with 1 as the coefficient of the term with the largest exponent) that satisfy the following conditions.

➡L138

(1) $f(x)$ is a linear function, and for any constant c, $\int_{-1}^{1} f(x)\cdot c\,dx = 0$.

[Sol] Let $f(x) = x + k$

$$\int_{-1}^{1} f(x)\cdot c\,dx = c\int_{-1}^{1}(x+k)dx = 2c\int_{0}^{1} kdx = 2c\Big[kx\Big]_0^1 = 2ck = 0$$

In order for the above equation to be true for any value of c, k must equal zero.
Therefore, $k = 0$
Thus, $\boldsymbol{f(x) = x}$

(2) $g(x)$ is a quadratic function, and for any linear function $h(x)$, $\int_{-1}^{1} g(x)\cdot h(x)dx = 0$.

[Sol] Let $g(x) = x^2 + mx + n$, and let $h(x) = sx + t$ (where $s \neq 0$)

$$\int_{-1}^{1} g(x)\cdot h(x)dx = \int_{-1}^{1}(x^2+mx+n)(sx+t)dx$$

$$= s\int_{-1}^{1}(x^3+mx^2+nx)dx + t\int_{-1}^{1}(x^2+mx+n)dx$$

$$= 2s\int_{0}^{1} mx^2dx + 2t\int_{0}^{1}(x^2+n)dx = 2s\Big[\frac{m}{3}x^3\Big]_0^1 + 2t\Big[\frac{1}{3}x^3+nx\Big]_0^1$$

$$= \frac{2m}{3}s + 2\left(\frac{1}{3}+n\right)t = 0$$

In order for the above equation to be true for any values of s and t, the following two equations must be true:

$$\frac{2m}{3} = 0, \quad 2\left(\frac{1}{3}+n\right) = 0$$

Therefore, $m = 0$, $n = -\frac{1}{3}$

Thus, $\boldsymbol{g(x) = x^2 - \frac{1}{3}}$

L 193b

2. A cubic function $f(x)$ passes through the origin, and the slope of the tangent at the origin is 1. Determine this function $f(x)$, so that $\int_0^1 f(x)\cdot g(x)dx=0$ is always true for any linear function $g(x)$.

➡L55, 138

[Sol] Since the cubic function passes through the origin,

let $f(x)=ax^3+bx^2+cx$

$f'(x)=3ax^2+2bx+c$

From $f'(0)=c$, $c=1$

Let $g(x)=px+q$ (where $p\neq 0$)

$$\int_0^1 f(x)\cdot g(x)dx=\int_0^1(ax^3+bx^2+x)(px+q)dx$$
$$=p\int_0^1(ax^4+bx^3+x^2)dx+q\int_0^1(ax^3+bx^2+x)dx$$
$$=p\left[\frac{a}{5}x^5+\frac{b}{4}x^4+\frac{1}{3}x^3\right]_0^1+q\left[\frac{a}{4}x^4+\frac{b}{3}x^3+\frac{1}{2}x^2\right]_0^1$$
$$=\left(\frac{a}{5}+\frac{b}{4}+\frac{1}{3}\right)p+\left(\frac{a}{4}+\frac{b}{3}+\frac{1}{2}\right)q=0$$

In order for the above equation to be true for any values of p and q, the following two equations must be true:

$\frac{a}{5}+\frac{b}{4}+\frac{1}{3}=0$ thus, $12a+15b=-20$ ⋯①

$\frac{a}{4}+\frac{b}{3}+\frac{1}{2}=0$ thus, $3a+4b=-6$ ⋯②

From ① and ②,

$a=\frac{10}{3}$, $b=-4$

Therefore, $\boldsymbol{f(x)=\frac{10}{3}x^3-4x^2+x}$

L 194a KUMON

Summary of Differention and Integration II

1. Given that the curves $y = x^2 + 2x + 1$ and $y = -3x^2 + ax + b$ touch at point $(1,\ 4)$:

➡L181

(1) Find the values of a and b.

[Sol] Let $f(x) = x^2 + 2x + 1$, and let $g(x) = -3x^2 + ax + b$

From $f'(x) = 2x + 2$, $\quad f'(1) = 4$

From $g'(x) = -6x + a$, $\quad g'(1) = -6 + a$

Letting $f'(1) = g'(1)$,

$4 = -6 + a$

$a = 10$

As $g(1) = -3 + 10 + b = 4$, then $b = -3$

Therefore, $\boldsymbol{a = 10,\ b = -3}$

(2) Find the area, S, enclosed by the two curves and the y-axis.

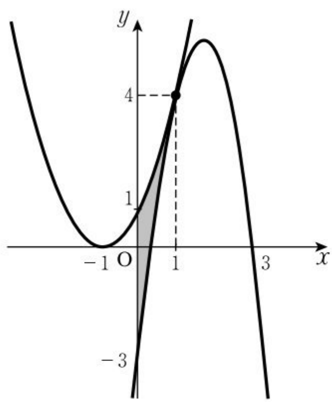
[Sol] $\boldsymbol{S} = \int_0^1 [(x^2 + 2x + 1) - (-3x^2 + 10x - 3)]dx$

$= \int_0^1 (4x^2 - 8x + 4)dx$

$= 4\left[\frac{1}{3}x^3 - x^2 + x\right]_0^1$

$= 4\left(\frac{1}{3} - 1 + 1\right)$

$= \boldsymbol{\frac{4}{3}}$



L 194b

2. Given that the curves $y=x^3-x$ and $y=x^2-a$ (where $a>0$) pass through point, P, and have a common tangent at point P:

(1) Find the value of a.

(2) Find the area, S, enclosed by the two curves.

[Sol] (1) Let $f(x)=x^3-x$, and let $g(x)=x^2-a$

$f'(x)=3x^2-1,\quad g'(x)=2x$

Let p be the x-coordinate of point P.

$f'(p)=3p^2-1,\quad g'(p)=2p$

Letting $f'(p)=g'(p),\quad 3p^2-1=2p$

$3p^2-2p-1=(3p+1)(p-1)=0,\quad p=-\dfrac{1}{3},\ 1$

When $p=-\dfrac{1}{3}$,

from $f\left(-\dfrac{1}{3}\right)=-\dfrac{1}{27}+\dfrac{1}{3}=\dfrac{8}{27}$, and $g\left(-\dfrac{1}{3}\right)=\dfrac{1}{9}-a$,

letting $f\left(-\dfrac{1}{3}\right)=g\left(-\dfrac{1}{3}\right),\quad \dfrac{1}{9}-a=\dfrac{8}{27}$

Therefore, $a=-\dfrac{5}{27}$

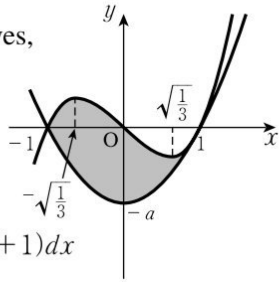
When $p=1$, from $f(1)=1-1=0$, and $g(1)=1-a$,
letting $f(1)=g(1),\quad 1-a=0$
Therefore, $a=1$

Since $a>0$, $a=-\dfrac{5}{27}$ is an extraneous solution.

Therefore, $\boldsymbol{a=1}$

(2) Finding the common points of the two curves,

$x^3-x=x^2-1$
$x^3-x^2-x+1=0$
$(x-1)^2(x+1)=0$
$x=-1,\ 1$



$\boldsymbol{S}=\displaystyle\int_{-1}^{1}[x^3-x-(x^2-1)]dx=2\int_{0}^{1}(-x^2+1)dx$

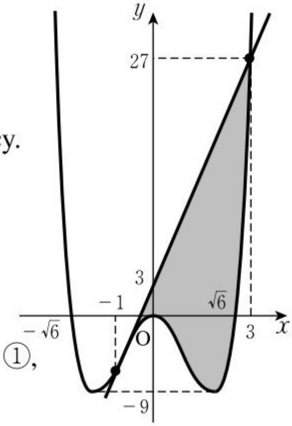
$=2\left[-\dfrac{1}{3}x^3+x\right]_0^1=2\left(-\dfrac{1}{3}+1\right)=\boldsymbol{\dfrac{4}{3}}$

L 195a KUMON

Summary of Differention and Integration II

1. Given the curve $y = x^4 - 6x^2$:

(1) Among the tangents passing through point $(0,\ 3)$, find the one that has a positive slope.



[Sol] Let $f(x) = x^4 - 6x^2$

$$f'(x) = 4x^3 - 12x$$

Let a be the x-coordinate of the point of tangency.

$$f'(a) = 4a^3 - 12a$$

Therefore, the equation of the tangent is:

$$y = (4a^3 - 12a)(x - a) + a^4 - 6a^2$$

$$= (4a^3 - 12a)x - 3a^4 + 6a^2 \quad \cdots ①$$

Substituting the coordinates of point $(0,\ 3)$ into ①,

$$3 = -3a^4 + 6a^2$$

$$3(a^4 - 2a^2 + 1) = 3(a^2 - 1)^2 = 0$$

$(a^2 - 1)^2 = (a+1)^2(a-1)^2 = 0$ Therefore, $a = \pm 1$

When $a = -1$, $f'(-1) = 8$

When $a = 1$, $f'(1) = -8 < 0$

This is an extraneous solution;
We are looking for a positive slope.

Therefore, $\boldsymbol{y = 8x + 3}$

(2) Find the area, *S*, enclosed by the curve and the tangent.

[Sol] From $x^4 - 6x^2 = 8x + 3$,

$$x^4 - 6x^2 - 8x - 3 = 0$$

$$(x-3)(x+1)^3 = 0$$

$$x = -1,\ 3$$

$$\boldsymbol{S} = \int_{-1}^{3} [8x + 3 - (x^4 - 6x^2)]\,dx$$

$$= \left[-\frac{1}{5}x^5 + 2x^3 + 4x^2 + 3x\right]_{-1}^{3}$$

$$= \left(-\frac{243}{5} + 54 + 36 + 9\right) - \left(\frac{1}{5} - 2 + 4 - 3\right) = \boldsymbol{\frac{256}{5}}$$

L 195b

2. Given the parabola $y = x^2 - 2x - 3$:

(1) Find the equations of the two lines that are tangent to the parabola at the x-axis.
➡L154

[Sol] From $x^2 - 2x - 3 = (x-3)(x+1) = 0$,

$x = -1,\ 3$

The lines cross the x-axis at points $(-1,\ 0)$ and $(3,\ 0)$.

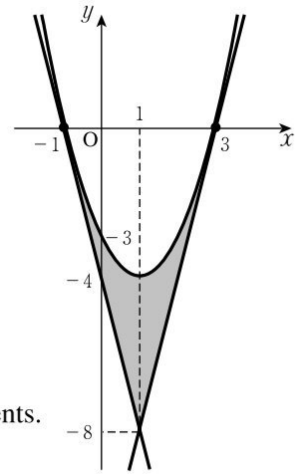
From $y' = 2x - 2$:

The tangent through $(-1,\ 0)$ is:

$\boldsymbol{y} = -4(x+1)\ \boldsymbol{= -4x - 4}$

The tangent through $(3,\ 0)$ is:

$\boldsymbol{y} = 4(x-3)\ \boldsymbol{= 4x - 12}$



(2) Find the point of intersection of the two tangents.

[Sol] From $-4x - 4 = 4x - 12$,

$8x - 8 = 0$

Therefore, $x = 1$, $y = -8$

The point of intersection is **(1, −8)**.

(3) Find the area, S, enclosed by the parabola and the two tangent lines from (1) above.

[Sol] $$\boldsymbol{S} = \int_{-1}^{1}[(x^2-2x-3)-(-4x-4)]dx + \int_{1}^{3}[(x^2-2x-3)-(4x-12)]dx$$

$$= \int_{-1}^{1}(x^2+2x+1)dx + \int_{1}^{3}(x^2-6x+9)dx$$

$$= 2\left[\frac{1}{3}x^3 + x\right]_0^1 + \left[\frac{1}{3}x^3 - 3x^2 + 9x\right]_1^3$$

$$= 2\left(\frac{1}{3}+1\right) + (9-27+27) - \left(\frac{1}{3} - 3 + 9\right)$$

$$\boldsymbol{= \frac{16}{3}}$$

L 196a KUMON

Summary of Differention and Integration II

1. When curve $C_1: y=x^3-x$ is translated a units along the x-axis ($a>0$), let C_2 be the resulting curve, $y=(x-a)^3-(x-a)$.

(1) Find the range of values of a for which curves C_1 and C_2 have at least one common point.

[Sol] $(x-a)^3-(x-a)=x^3-3ax^2+(3a^2-1)x-a^3+a$

Since the curves C_1 and C_2 have a common point, at that point,

$$x^3-x=x^3-3ax^2+(3a^2-1)x-a^3+a$$

Solving this equation,

$$3ax^2-3a^2x+a^3-a=0$$

Since $a\neq 0$, $\quad 3x^2-3ax+a^2-1=0 \quad \cdots①$

Calculating the discriminant of ①,

$$D=9a^2-4\cdot 3(a^2-1)\geqq 0$$
$$3a^2-12\leqq 0$$
$$3(a+2)(a-2)\leqq 0$$

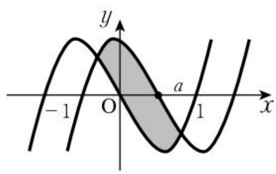
Since $a>0$, $\quad \boldsymbol{0<a\leqq 2}$

(2) Using the result of (1), express the area, S, enclosed by the two curves C_1 and C_2, in terms of a.

[Sol] From $(x-a)^3-(x-a)-(x^3-x)=-3ax^2+3a^2x-a^3+a$,

let the roots of ① be α and β (where $\alpha<\beta$)

$$\alpha+\beta=a,\ \alpha\beta=\frac{a^2-1}{3}$$



$$\boldsymbol{S}=\int_\alpha^\beta(-3ax^2+3a^2x-a^3+a)dx$$

$$=-3a\int_\alpha^\beta\left(x^2-ax+\frac{a^2-1}{3}\right)dx$$

$$=\frac{a}{2}(\beta-\alpha)^3$$

Use the formula on L124b.

$$=\frac{a}{2}\left[\sqrt{(\alpha+\beta)^2-4\alpha\beta}\right]^3=\frac{a}{2}\left[\sqrt{a^2-\frac{4}{3}(a^2-1)}\right]^3$$

$$\boldsymbol{=\frac{a}{2}\left(\sqrt{\frac{4-a^2}{3}}\right)^3}$$

L 196b

2. Find the area, S, enclosed by the following curve and line:

$$y = |\ x^2 - x - 2\ |, \quad y = x + 1$$

➡L38

[Sol] $x^2 - x - 2 = (x+1)(x-2)$

When $x \leqslant -1$ or $x \geqslant 2$,

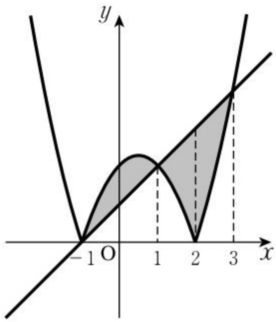
$$|\ x^2 - x - 2\ | = x^2 - x - 2$$

From $x^2 - x - 2 = x + 1$,

$$x^2 - 2x - 3 = 0$$

$$(x-3)(x+1) = 0$$

Therefore, $x = -1,\ 3$



When $-1 \leqslant x \leqslant 2$,

$$|\ x^2 - x - 2\ | = -x^2 + x + 2$$

From $-x^2 + x + 2 = x + 1$,

$$x^2 - 1 = 0$$

Therefore, $x = -1,\ 1$

$$\boldsymbol{S} = \int_{-1}^{1} [(-x^2 + x + 2) - (x+1)]\,dx + \int_{1}^{2} [(x+1) - (-x^2 + x + 2)]\,dx$$

$$+ \int_{2}^{3} [(x+1) - (x^2 - x - 2)]\,dx$$

$$= \int_{-1}^{1} (-x^2 + 1)\,dx + \int_{1}^{2} (x^2 - 1)\,dx + \int_{2}^{3} (-x^2 + 2x + 3)\,dx$$

$$= 2\left[-\frac{1}{3}x^3 + x\right]_0^1 + \left[\frac{1}{3}x^3 - x\right]_1^2 + \left[-\frac{1}{3}x^3 + x^2 + 3x\right]_2^3$$

$$= 2\left(-\frac{1}{3} + 1\right) + \left[\left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - 1\right)\right]$$

$$+ \left[(-9 + 9 + 9) - \left(-\frac{8}{3} + 4 + 6\right)\right]$$

$$= \boldsymbol{\frac{13}{3}}$$

L 197a KUMON

Summary of Differention and Integration II

1. Given the curves $C_1 : y = x^2$ and $C_2 : y = (x-2)^2 + 2$, complete the exercises on side a and side b.

(1) Curve C_1 has a tangent at point $(p,\ p^2)$. The same line is also tangent to curve C_2. Find the equation of the tangent.

[Sol] Let $f(x) = x^2$

$$f'(x) = 2x$$

Using the coordinates $(p,\ p^2)$, and the gradient,

the equation of the tangent is

$$y = 2p(x-p) + p^2 = 2px - p^2 \quad \cdots ①$$

Setting ① equal to C_2,

$$2px - p^2 = (x-2)^2 + 2$$

$$x^2 - 2(p+2)x + 6 + p^2 = 0$$

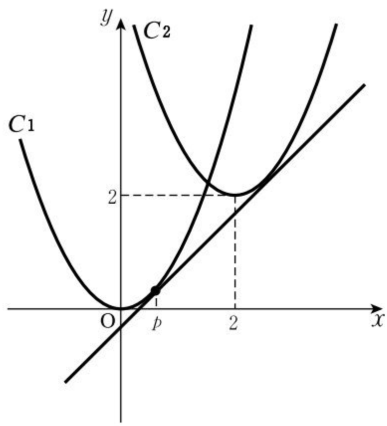
$$\frac{D}{4} = (p+2)^2 - (6+p^2)$$

$$= 4p - 2 = 0$$

Therefore, $p = \dfrac{1}{2} \quad \cdots ②$

Substituting ② into ①,

$$\boldsymbol{y = x - \frac{1}{4}}$$



L 197b

(2) Let a be the x-coordinate of the point where C_1 and C_2 intersect. Find the value of a.

[Sol] $a^2=(a-2)^2+2$

$4a-6=0$

Therefore, $\boldsymbol{a=\frac{3}{2}}$

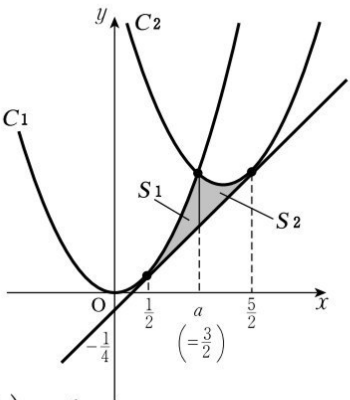
(3) Let S_1 be the area enclosed by C_1, the tangent from (1), and line $x=a$. Let S_2 be the area enclosed by C_2, the tangent from (1), and line $x=a$. Find the ratio of the areas $S_1 : S_2$.

[Sol] Finding the point of tangency of C_2 and the tangent from (1),

$$(x-2)^2+2=x-\frac{1}{4}$$

$$(2x-5)^2=0$$

$$x=\frac{5}{2}$$



$$S_1=\int_{\frac{1}{2}}^{\frac{3}{2}}\left[x^2-\left(x-\frac{1}{4}\right)\right]dx$$

$$=\left[\frac{1}{3}x^3-\frac{1}{2}x^2+\frac{1}{4}x\right]_{\frac{1}{2}}^{\frac{3}{2}}$$

$$=\left(\frac{9}{8}-\frac{9}{8}+\frac{3}{8}\right)-\left(\frac{1}{24}-\frac{1}{8}+\frac{1}{8}\right)=\frac{1}{3}$$

$$S_2=\int_{\frac{3}{2}}^{\frac{5}{2}}\left[x^2-4x+6-\left(x-\frac{1}{4}\right)\right]dx$$

$$=\left[\frac{1}{3}x^3-\frac{5}{2}x^2+\frac{25}{4}x\right]_{\frac{3}{2}}^{\frac{5}{2}}$$

$$=\left(\frac{125}{24}-\frac{125}{8}+\frac{125}{8}\right)-\left(\frac{9}{8}-\frac{45}{8}+\frac{75}{8}\right)$$

$$=\frac{125}{24}-\frac{39}{8}=\frac{1}{3}$$

Therefore, $S_1 : S_2=\frac{1}{3} : \frac{1}{3}=\boldsymbol{1 : 1}$

L 198a KUMON

Summary of Differention and Integration II

1. The two curves $C_1 : y = x^2$ and $C_2 : y = -2x^2 + ax + b$ share a common tangent with gradient 4, at 1 point.

(1) Find the point of tangency, and the values of a and b.

[Sol] Let $f(x) = x^2$, and let $g(x) = -2x^2 + ax + b$

$f'(x) = 2x$

From $2x = 4$, $x = 2$ and $f(2) = 4$

Therefore, the point of tangency is $\mathbf{(2,\ 4)}$.

$g'(x) = -4x + a, \quad g'(2) = -8 + a$

From $-8 + a = 4, \quad a = 12$

From $g(2) = -8 + 2a + b = 4, \quad b = -12$

Therefore, $\boldsymbol{a = 12}$ and $\boldsymbol{b = -12}$

(2) Find the equation of the line with gradient -1 passing through the point of tangency of C_1 and C_2.

[Sol] $y = -1 \cdot (x - 2) + 4$

Therefore, $\boldsymbol{y = -x + 6}$

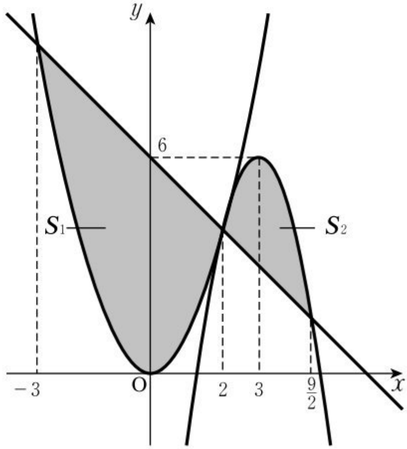
L 198b

(3) Let S_1 be the area enclosed by the line from (2) and C_1.
Let S_2 be the area enclosed by the line from (2) and C_2.
Find the ratio of the areas $S_1 : S_2$.

[Sol] Finding the points of intersection of C_1 and $y=-x+6$:

$$x^2=-x+6$$
$$x^2+x-6=0$$
$$(x+3)(x-2)=0$$

Therefore, $x=-3,\ 2$



$$S_1=\int_{-3}^{2}(-x+6-x^2)dx$$
$$=-\int_{-3}^{2}(x+3)(x-2)dx$$
$$=\frac{1}{6}(2+3)^3$$
$$=\frac{125}{6}$$

Finding the points of intersection of C_2 and $y=-x+6$:

$$-2x^2+12x-12=-x+6$$
$$2x^2-13x+18=0$$
$$(2x-9)(x-2)=0$$

Therefore, $x=2,\ \frac{9}{2}$

$$S_2=\int_{2}^{\frac{9}{2}}[-2x^2+12x-12-(-x+6)]dx$$
$$=-2\int_{2}^{\frac{9}{2}}(x-2)\left(x-\frac{9}{2}\right)dx$$
$$=\frac{1}{3}\left(\frac{9}{2}-2\right)^3$$
$$=\frac{125}{24}$$

Therefore, $S_1 : S_2=\frac{125}{6} : \frac{125}{24}=\mathbf{4 : 1}$

L 199a KUMON

Summary of Differention and Integration II

1. Points A$(2,\ 2)$ and B$(-1,\ -1)$ are both on the parabola $y=x^2-2$. Placing point P$(p,\ p^2-2)$ on the parabola between points A and B, complete the exercises on side a and side b.

(1) Express the area of $\triangle$ABP in terms of p.

[Sol] Let Q be the point of intersection of line AB and line $x=p$.

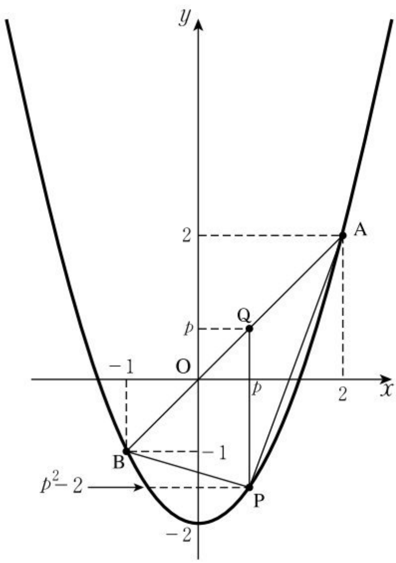
Since the gradient of AB is 1, and since the line passes through the origin, the equation of line AB is:

$$y=x$$

The coordinates of point Q are $(p,\ p)$.

$$PQ=p-(p^2-2)=-p^2+p+2$$

If we split $\triangle$ABP into 2 triangles, $\triangle$PQA and $\triangle$PQB, with PQ as the base of each triangle, the heights are $2-p$ and $p+1$, respectively.



$$\triangle PQA=\frac{1}{2}(-p^2+p+2)(2-p)$$

$$\triangle PQB=\frac{1}{2}(-p^2+p+2)(p+1)$$

Therefore,

$$\triangle \mathbf{ABP}=\triangle PQA+\triangle PQB$$

$$=\frac{1}{2}(-p^2+p+2)(2-p+p+1)$$

$$\boldsymbol{=-\frac{3}{2}(p^2-p-2)}$$

L 199b

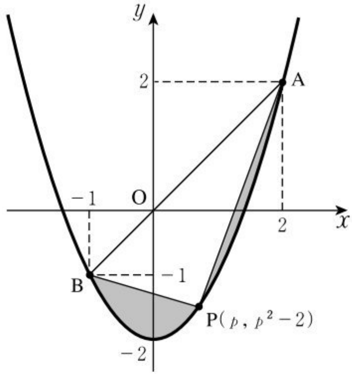
(2) Find the value of p for which the area, S, of the region enclosed by AP, BP and the parabola will be at a minimum value.

[Sol] Let S_1 be the area of the region enclosed by the parabola and line AB.

$$S_1 = \int_{-1}^{2} [x-(x^2-2)]\,dx$$

$$= -\int_{-1}^{2}(x+1)(x-2)\,dx$$

$$= \frac{1}{6}(2+1)^3 = \frac{9}{2}$$



$$S = S_1 - \triangle ABP$$

$$= \frac{9}{2} + \frac{3}{2}(p^2-p-2)$$

$$= \frac{3}{2}(p^2-p+1)$$

$$= \frac{3}{2}\left(p-\frac{1}{2}\right)^2 + \frac{9}{8}$$

Therefore, the area, S, is at a minimum when $\boldsymbol{p = \frac{1}{2}}$.

[Refer to the next page for the alternative solution.]

L 199a KUMON

(Alternative Solution)

1. Points A$(2,\ 2)$ and B$(-1,\ -1)$ are both on the parabola $y=x^2-2$. Placing point P$(p,\ p^2-2)$ on the parabola between points A and B, complete the exercises on side a and side b.

(1) Express the area of $\triangle$ABP in terms of p.

[Sol] As $p \neq 2$, the equation of line AP is:

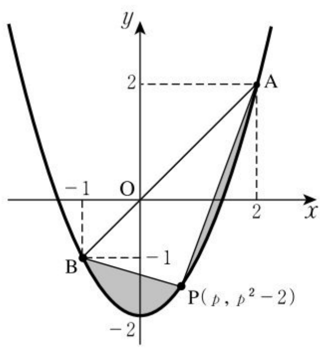
$$y=\frac{(p^2-2)-2}{p-2}(x-2)+2=(p+2)x-2p-2$$

As $p \neq -1$, the equation of line BP is:

$$y=\frac{(p^2-2)+1}{p+1}(x+1)-1=(p-1)x+p-2$$

Let S_1 be the area enclosed by the parabola and line AP.

$$S_1=\int_p^2[(p+2)x-2p-2-(x^2-2)]\,dx$$
$$=-\int_p^2[x^2-(p+2)x+2p]\,dx$$
$$=-\int_p^2(x-p)(x-2)\,dx=\frac{1}{6}(2-p)^3$$



Let S_2 be the area enclosed by the parabola and line BP.

$$S_2=\int_{-1}^p[(p-1)x+p-2-(x^2-2)]\,dx$$
$$=-\int_{-1}^p[x^2-(p-1)x-p]\,dx$$
$$=-\int_{-1}^p(x+1)(x-p)\,dx=\frac{1}{6}(p+1)^3$$

Let S be the area enclosed by the parabola and lines AP and BP.

Therefore,

$$S=S_1+S_2=\frac{1}{6}(2-p)^3+\frac{1}{6}(p+1)^3=\frac{3}{2}(p^2-p+1)$$

The equation of line AB is $y=x$.

Let S_3 be the area enclosed by the parabola and line AB.

$$S_3=\int_{-1}^2[x-(x^2-2)]\,dx=-\int_{-1}^2(x+1)(x-2)\,dx=\frac{1}{6}(2+1)^3=\frac{9}{2}$$

Therefore,

$$\triangle\mathrm{ABP}=\frac{9}{2}-\frac{3}{2}(p^2-p+1)=\boldsymbol{-\frac{3}{2}(p^2-p-2)}$$

L 199b

(2) Find the value of p for which the area, S, enclosed by AP, BP and the parabola will be at a minimum value.

[Sol] $S = S_1 + S_2$

$$= \frac{3}{2}(p^2 - p + 1)$$

$$= \frac{3}{2}\left(p - \frac{1}{2}\right)^2 + \frac{9}{8}$$

Therefore, the area, S, is at a minimum when $\boldsymbol{p = \frac{1}{2}}$.

L 200a KUMON

Summary of Differention and Integration II

1. Given that the parabola $y=x^2+1$ has a tangent at point $\mathrm{P}(a,\ a^2+1)$, find the area, S, enclosed by the tangent and parabola $y=x^2$.

[Sol] Let $f(x)=x^2+1$

From $f'(x)=2x$, $\quad f'(a)=2a$

The equation of the tangent is:

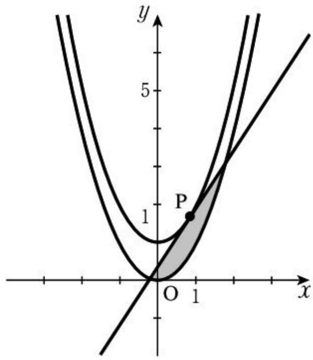
$$y=2a(x-a)+a^2+1=2ax-a^2+1$$

Setting this equal to $y=x^2$,

to find the points of intersection,

$$x^2=2ax-a^2+1$$

$$x^2-2ax+a^2-1=0$$



Solving the equation $x^2-2ax+a^2-1=0$,

let α and β (where $\alpha<\beta$) be the roots, so that

$$\alpha+\beta=2a, \quad \alpha\beta=a^2-1$$

Therefore,

$$\boldsymbol{S}=\int_\alpha^\beta(2ax-a^2+1-x^2)dx$$

$$=-\int_\alpha^\beta(x^2-2ax+a^2-1)dx$$

$$=\frac{1}{6}(\beta-\alpha)^3$$

$$=\frac{1}{6}\left[\sqrt{(\alpha+\beta)^2-4\alpha\beta}\right]^3$$

Use the formula on L124b.

$$=\frac{1}{6}\left(\sqrt{4a^2-4a^2+4}\right)^3$$

$$\boldsymbol{=\frac{4}{3}}$$

L 200b

2. The parabola $y=-x^2$ has two tangents which connect at point P, located on line $y=-x+1$. Find the x-coordinate of point P for which the area, *S*, enclosed by the line connecting the two tangent points and the parabola, is at a minimum.

[Sol] Let the coordinates of point P be $(a,\ -a+1)$.

Let the points of tangency to $y=-x^2$ be $(t,\ -t^2)$.

The tangent is then:

$$y=-2t(x-t)-t^2=-2tx+t^2 \quad \cdots ①$$

Substituting the coordinates of P$(a,\ -a+1)$ into ①,

$$-a+1=-2ta+t^2$$

$$t^2-2at+a-1=0 \quad \cdots ②$$

If α and β (where $\alpha<\beta$) are the x-coordinates of the points of tangency between the two tangent lines and the parabola, then α and β are the roots of ②.

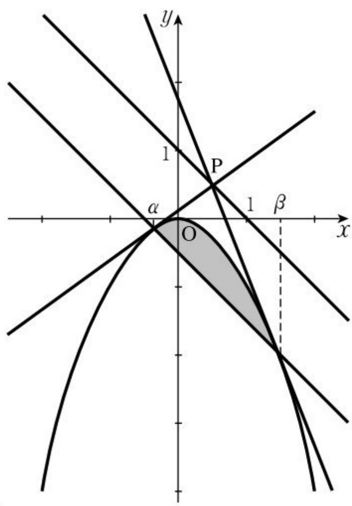
Therefore $\alpha+\beta=2a,\ \alpha\beta=a-1$

The equation of the line which passes through the two points of tangency, $(\alpha,\ -\alpha^2)$ and $(\beta,\ -\beta^2)$, is:

$$y=\frac{-\alpha^2+\beta^2}{\alpha-\beta}(x-\alpha)-\alpha^2=-(\alpha+\beta)x+\alpha\beta=-2ax+a-1$$

Therefore,

$$\begin{aligned}
S&=\int_\alpha^\beta[-x^2-(-2ax+a-1)]dx\\
&=-\int_\alpha^\beta(x^2-2ax+a-1)dx\\
&=\frac{1}{6}(\beta-\alpha)^3\\
&=\frac{1}{6}\left[\sqrt{(\alpha+\beta)^2-4\alpha\beta}\right]^3\\
&=\frac{1}{6}\left[\sqrt{4a^2-4(a-1)}\right]^3\\
&=\frac{1}{6}\left[\sqrt{4\left(a-\frac{1}{2}\right)^2+3}\right]^3
\end{aligned}$$



When $a=\frac{1}{2}$, the area, *S*, is at a minimum.

Therefore, the x-coordinate of point P is $\mathbf{\frac{1}{2}}$.